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CATEGORICAL SYSTEM THEORY

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This is an investigation of natural systems from the standpoint of the mathematical theory of categories. It examines the relationships which exist between different descriptions through measurement of observables and dynamical interactions. We begin with a category theory of formal systems with observables, and then proceed to a category theory of dynamical systems. The two categories are then combined to represent natural systems. Topological considerations enter in the study of stability and bifurcation phenomena. Special emphasis is placed on natural systems which model biological processes. The categorical system theory developed is applied to the analysis of several biological problems and biological system theories.

1. Introduction. In his book *The Scientific Outlook* (1931), Russell described the 'scientific process' as composed of

> "three main stages; the first consists in observing the significant facts; the second in arriving at a hypothesis which, if it is true, would account for these facts; the third in deducing from this hypothesis further consequences which can be tested by observation."

So according to Russell, the act of *observation* is basic to science. Intuitively, the notion of *observable* is attached to that of a concrete procedure for determining the value assumed by the observable of a system at a specific time. The crucial ingredient of any such procedure is a measuring instrument, which forms the basis both for our knowledge of the physical world and for our formulation of models which organize this knowledge and allow us to predict and control.

Rosen (1978), the single greatest influence in the development of this work, provides a comprehensive theory of observables and the descriptions arising from them. The theory is then applicable to any situation in which objects of interest are labelled by definite mapping processes, measurement in physics, pattern recognition, discrimination or classification. All of these diverse situations share a common character: namely, the generation of numbers (or other kinds of invariants) which serve to label the processes with which they are associated, such that processes are considered 'the same' if and only if they bear the same label. This leads to the idea of observable-induced equivalence relations (Section 3).

It is suggested in Rosen (1978) that a formal treatment of systems with observables using category theory would be a fruitful undertaking. The engagement in this problem marked the beginning of my dissertation. But why stop at systems with observables? Since the process of observation ultimately rests on the capacity of a given system to induce a dynamics (i.e. a change of state) in a measuring instrument (alias meter, recognizer, discriminator, classifier, etc.), it seems natural to consider systems with dynamics as well.

There is a reciprocity that exists in general dynamical interactions between systems. The process of measurement can be considered as a reciprocal induction of dynamics in both the system being measured and the system which measures. Then the basic problem in the analysis becomes this: to determine the observables through which a particular given dynamics is taking place, to specify the subsystems to which these observables belong and to identify the manner in which each of these subsystems is causing the others to change states.

The concepts of linkage of observables, stability and bifurcation, and their connections with dynamics are also treated in this paper in the context of the category of natural systems, an amalgamation of states, observables and dynamics. The categories constructed have curious links with diverse branches of mathematics from topology to Galois theory, and provide a natural setting to discuss the modelling relation. There are also many biological implications of categorical system theory. Among these, cellular dynamics, growth and aging and Rashevsky's (1972) organismic sets are given as examples. Other examples will appear in subsequent papers.

Category theory is the mathematical tool used in this work. The basic definitions, treated in any one of the standard texts on the subject (e.g. Mac Lane, 1971), are assumed.

2. Propositions. Throughout this study we will be dealing with three basic concepts: *system, state* and *observable*. Intuitively, a *system* is some part of the real world which is our object to study; a *state* is a specification of what our system is like at a particular time; and an *observable* of the system is some characteristic of the system which can, at least in principle, be measured. In other words, an observable of a system is a quantity which can induce dynamics in some appropriate meter.

These three basic concepts are interrelated via two fundamental propositions which we shall take as axioms in all of what follows:

PROPOSITION 1. *The only meaningful physical events which occur in the world are those represented by the evaluation of observables on states.*

PROPOSITION 2. *Every observable can be regarded as a mapping from states to real numbers.*

Scientific activity usually starts with the collection of observable phenomena within a given field. The significance of Proposition 1 lies in the word *meaningful*—because our information on a system is derived from what we can observe, hence what we cannot observe (in the generalized sense of creating percepts in our brains) will not be meaningful to us. This points also to the fact that as our means of observation increase, the more 'alternate descriptions' we have on a system, the more insights in how and why things work we will have.

To see why Proposition 2 makes sense, let us first quote Eddington from his *Fundamental Theory* (1949):

> "The whole subject matter of exact science consists of pointer readings and similar indications; whatever quantity we say we are 'observing', the actual procedure nearly always ends in reading the position of some kind of indicator on a graduated scale or its equivalent."

In science the most common form of questioning nature is through experiments, and the answer often comes in *numerical* readings.

It may well be that from the point of view of scientific enquiry, the only way we can handle any entity is through the numbers associated with its measurement, which in turn is defined through the measuring instrument, the meter. This, in particular, is the view of Einstein on *time*. Such a definition of time—as something dependent of a measuring instrument—is called an *operational definition*. Borrowing the terminology, our two propositions are then saying that all our observables are operationally defined and all our meters are real-valued.

It must be realized, however, that the operational definition of scientific entities will enable us to investigate only certain aspects of science. Other questions are in principle unanswerable, i.e. they will not yield a measurement that gives an answer to the question. Such questions are then meaningless in the context of scientific investigation. But the study of those aspects amenable to science based on operational definitions is enough to keep us busy forever. After all, the study of a model of the *whole* of nature belongs to metaphysics, not science.

3. Observables and Equivalence Relations. Let us consider the prototype situation, in which we have a set S of states and a real-valued function $f : S \to \mathbb{R}$, which represents an observable. f induces an equivalence relation R_f on S defined by

$$s_1 R_f s_2 \text{ iff } f(s_1) = f(s_2).$$

Clearly the quotient set S/R_f is in one-to-one correspondence with the spectrum $f(S)$. In general the observable f conveys limited information about its domain S because by definition it cannot distinguish between states lying in the same equivalence class, and the set of states of our system would appear to be S/R_f. This is why 'alternate descriptions' of a system are important: the more observables we have, the more information we have on S.

On the other hand, the equivalence relations on S induce an equivalence $\sim$ on $\mathbb{R}^S$, the set of all real-valued functions on S, as follows. For $f, g \in \mathbb{R}^S$, define $f \sim g$ iff $R_f = R_g$, i.e. iff $f(s_1) = f(s_2)$ is equivalent to $g(s_1) = g(s_2)$. Two equivalent observables 'convey the same information' about the elements of S; therefore we cannot distinguish between elements of S by employing equivalent observables. Note, however, that $f \sim g$ only means that $S/R_f = S/R_g$, i.e. there is a one-to-one correspondence between $f(S)$ and $g(S)$ and there need be no relation whatsoever between the values of $f(s)$ and $g(s)$ for $s \in S$. In particular, $|f(s_1) - f(s_2)|$ small does not imply $|g(s_1) - g(s_2)|$ small. Thus if we are considering the metric aspects of observables (which we will do in later sections) we cannot pass to the equivalence classes in $\mathbb{R}^S/\sim$. But when we are only interested in the *reduced states* in S/R_f and not the specific values of $f(S)$, it is more convenient to consider observables as elements of $\mathbb{R}^S/\sim$. One reason for this is that $\mathbb{R}^S/\sim$ is a partially ordered set under the relation *refinement*. [Define *f refines g*, $f \leqslant g$, if $f(s_1) = f(s_2)$ implies $g(s_1) = g(s_2)$, i.e. if $R_f \subset R_g$ for $f, g \in \mathbb{R}^S$. Then it is clear that $\leqslant$ is reflexive and transitive on $\mathbb{R}^S$ and hence is a preorder. But $f \leqslant g$ and $g \leqslant f$ only implies $R_f = R_g$ (i.e. $f \sim g$) and not $f = g$. So to make $\leqslant$ antisymmetric one passes onto $\mathbb{R}^S/\sim$. Note that $f \leqslant g$ and $g \leqslant f$ iff $f \sim g$.] It is not uncommon in mathematics to consider equivalence classes of functions instead of the functions themselves—the L^p spaces, for example, are equivalence classes of functions with $f \sim g$ iff $f = g$ almost everywhere.

4. The Category **S**. We will now undertake a formal treatment of systems, considered as abstract mathematical objects. We will be studying the properties of the category **S** of (formal) systems and comparing them to those of the category **Ens** of sets.

An object of **S**, a formal system, shall consist of a pair (S, F), where S is a set and F is a set of real-valued functions on S. The elements of S are the states and the elements of F are the observables of the formal system. We shall always assume $0 \in F$ (where 0 is the 'zero function' on S sending all states to the number 0), although for brevity we may sometimes omit 0 when we list the elements of F in specific examples. Thus we have F non-empty in order to avoid 'empty set pathologies'. The observable 0 is simply 'identifying the states in S'. Note that $(0)_\sim$ is the greatest element in the partially ordered set $(\mathbb{R}^S/\sim, \leqslant)$.

An **S**-morphism $\phi \in \mathbf{S}((S_1, F_1), (S_2, F_2))$ is a pair of functions $\phi \in \mathbf{Ens}(S_1, S_2)$ and $\phi \in \mathbf{Ens}(F_1, F_2)$ such that for all $f \in F_1$ for all $s, s' \in S_1, f(s) = f(s')$ implies $(\phi f)(\phi s) = (\phi f)(\phi s')$, i.e. $s \, R_f \, s'$ implies $\phi s \, R_{\phi f} \, \phi s'$.

Note that this 'compatibility' condition is equivalent to saying for all $G \subset F_1$ for all $s, s' \in S_1$, $s \, R_G \, s'$ implies $\phi s \, R_{\phi G} \, \phi s'$, where $R_G = \cap R_f$: $f \in G$ [hence $s \, R_G \, s'$ iff for all $f \in G$ $f(s) = f(s')$] and $\phi G = \{\phi f : f \in G\} \subset F_2$. This means that for all $G \subset F_1$, ϕ can be considered as a mapping from S_1/R_G to $S_2/R_{\phi G}$.

We always define $\phi 0 = 0$. This is compatible because clearly $0s = 0s'$ implies $0(\phi s) = 0(\phi s')$. Note also that for any observable f, the assignment $\phi f = 0$ is acceptable.

Define the identity $1_{(S,F)} \in \mathbf{S}((S, F), (S, F))$ by for all $s \in S$ $s \mapsto s$ and for all $f \in F$ $f \mapsto f$. (Thus for all $G \subset F$, $G \mapsto G$.) Then clearly $1_{(S,F)}$ satisfies the compatibility condition.

Define composition of morphisms in **S** as simultaneously the compositions on the states and on the observables, i.e. if ϕ: $(S_1, F_1) \to (S_2, F_2)$ and ψ: $(S_2, F_2) \to (S_3, F_3)$, define $\psi \circ \phi$: $(S_1, F_1) \to (S_3, F_3)$ by for every $s \in S_1$ $\psi \circ \phi(s) = \psi(\phi(s))$ and for every $f \in F_1$ $\psi \circ \phi(f) = \psi(\phi f)$. Note for $f \in F_1$ and $s, s' \in S_1$, $s \, R_f \, s'$ implies $\phi s \, R_{\phi f} \, \phi s'$, which in turn implies $\psi(\phi s) \, R_{\psi(\phi f)} \, \psi(\phi s')$; so $\psi \circ \phi$ satisfies the compatibility condition.

Clearly composition so defined is associative, and for ϕ: $(S_1, F_1) \to (S_2, F_2)$, $1_{(S_2,F_2)} \circ \phi = \phi = \phi \circ 1_{(S_1,F_1)}$.

If ϕ: $(S_1, F_1) \to (S_2, F_2)$ and ψ: $(S_2, F_2) \to (S_1, F_1)$ are such that $\psi \circ \phi = 1_{(S_1,F_1)}$ and $\phi \circ \psi = 1_{(S_2,F_2)}$, then it is easy to see that ϕ: $S_1 \to S_2$ and ϕ: $F_1 \to F_2$ must be bijections (**Ens**-isomorphisms) and that for $f \in F_1$ and $s, s' \in S_1$, $f(s) = f(s')$ iff $(\phi f)(\phi s) = (\phi f)(\phi s')$, i.e. for every $G \subset F_1$ $S_1/R_G = S_2/R_{\phi G}$.

Thus isomorphic systems are abstractly the same in the sense that there is a 'dictionary' (one-to-one correspondence) between the states and between the observables inducing the 'same' equivalence relations on the states. In particular, if F and G are two sets of observables on S and there is a bijection ϕ: $F \to G$ such that for all $f \in F$ $f \sim \phi f$, then the two systems (S, F) and (S, G) are isomorphic with the **S**-isomorphism $1_S : S \to S$, ϕ: $F \to G$. Since categorical constructions are only unique up to isomorphism, in the category **S** all constructions (S, F) are only 'unique' up to $\sim$-equivalent observables (i.e. one can always replace F by an $\sim$-equivalent set of observables G in the above sense), even when the set of states S is held fixed.

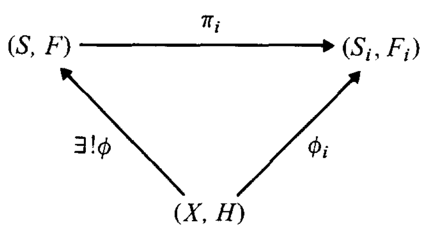
*5. **S**-Products.* Products in the category **S** do not always exist. For a family $\{(S_i, F_i): i \in I\}$ the product should be $(S, F) = \Pi(S_j, F_j)$: $j \in I$, with an I-tuple of **S**-morphisms of the form π_i: $(S, F) \to (S_i, F_i)$. S is defined as the cartesian product ΠS_j of the sets of states. F is defined as the cartesian product ΠF_j of the sets of observables interpreted as follows: for the observable $(f_j: j \in I)$ in F, it is a mapping from S to $\mathbb{R}^I$ defined by

$$(f_j: j \in I)(s_j: j \in I) = (f_j(s_j): j \in I).$$

But **S**-objects can only have real-valued observables so the mapping $f = (f_j: j \in I): S \to \mathbb{R}^I$ must be *represented* by an equivalent mapping from S to $\mathbb{R}$ (such that S/R_f remains the same). In other words, we need a one-to-one map from $(\mathbb{R}^I)^S$ to $\mathbb{R}^S$ which preserves the equivalence relations on S induced by the observables, or what is equivalent, an injection from $\mathbb{R}^I$ to $\mathbb{R}$. An injection from $\mathbb{R}^I$ to $\mathbb{R}$ only exists when the cardinality of I is less than that of $\mathbb{R}$; thus **S** only has countable (including finite) products. Note, however, that the **S**-product is independent of the choice of the injection from $\mathbb{R}^I$ to $\mathbb{R}$ because the effect of changing the injection is simply a switch from F to an $\sim$-equivalent set G, and (S, F) and (S, G) are isomorphic. Different choices of the injection define different **S**-isomorphism class representatives of the product.

The projections are obviously defined by $\pi_i((s_j: j \in I)) = s_i$ and $\pi_i((f_j: j \in I)) = f_i$, and it is easily checked that the π_i are indeed **S**-morphisms.

To see that (S, F) is a product, consider an **S**-object (X, H) equipped with an I-tuple of **S**-morphisms $\phi_i: (X, H) \to (S_i, F_i)$. Then



we can define $\phi: (X, H) \to (S, F)$ by for $x \in X$ $\phi(x) = (\phi_j(x): j \in I)$, and for $h \in H$ $\phi(h) = (\phi_j(h): j \in I)$, where again $\phi(h)$ is to be represented by a mapping from S to $\mathbb{R}$ via the injection from $\mathbb{R}^I$ to $\mathbb{R}$. It is clear that ϕ is the unique map which makes the diagram commute. (But ϕ is, of course, dependent on the choice of the injection from $\mathbb{R}^I$ to $\mathbb{R}$ which determines the product (S, F).) To see that ϕ is an **S**-morphism, let $h \in H$ and $x, x' \in X$ be such that $h(x) = h(x')$. Then for each $i \in I$ $(\phi_i h)(\phi_i x) = (\phi_i h)(\phi_i x')$ because each ϕ_i is an **S**-morphism. Whence by definition $(\phi h)(\phi x) = (\phi h)(\phi x')$. Thus $x \; R_h \; x'$ implies $\phi x \; R_{\phi h} \; \phi x'$. So (S, F) with the π_is satisfy the universal property in the definition of a categorical product.

The final object in **S** is $(1, \{0\})$, where 1 is the singleton set, the final **Ens**-object. The unique **S**-morphism from any system to $(1, \{0\})$ is clearly the one which sends all states to 1 and all observables to 0.

6. An Application: Linkage. Let S be a set of states and $f, g \in \mathbb{R}^S$ be observables. Let $p_f: S \to S/R_f$ and $p_g: S \to S/R_g$ be the natural quotient maps. For $(s)_f \in S/R_f$ consider the set of R_g-classes which intersect $(s)_f$, i.e. the set

$$p_g p_f^{-1}(s)_f = \{(s')_g : f(s') = f(s)\} = \{(s')_g : (s')_g \cap (s)_f \neq \emptyset\}.$$

Then we define:

(i) g is *totally linked* to f at $(s)_f$ if the above set consists of a single R_g-class [necessarily $(s)_g$; i.e. $f(s) = f(s')$ implies $g(s) = g(s')$].
(ii) g is *partially linked* to f at $(s)_f$ if this set consists of more than one R_g-class, but is not all of S/R_g.
(iii) g is *unlinked* to f at $(s)_f$ if this set is S/R_g.

Also, we say g is *totally linked* to f if it is totally linked at each $(s)_f$ and g is *unlined* to f if it is unlinked at each $(s)_f$.

From the above definition it is immediate that g is totally linked to f iff R_f refines R_g, which is equivalent to the existence of an **S**-morphism from $(S, \{f\})$ to $(S, \{g\})$, which sends each $s \in S$ to itself and sends f to g, for the latter statement means precisely that $f(s) = f(s')$ implies $g(s) = g(s')$.

For a set of states S equipped with two distinct observables f and g there is another equivalence relation on S other than R_f and R_g which is of interest—namely, the intersection $R_{fg} = R_f \cap R_g$. The relation R_{fg} is defined by $s\ R_{fg}\ s'$ iff $f(s) = f(s')$ *and* $g(s) = g(s')$. Note that there may not be an observable of S which generates the equivalence relation R_{fg}, i.e. although mathematically there exists $h \in \mathbb{R}^S$ such that $R_{fg} = R_h$, the set of all possible observables of S, as a representation of a natural system, may not be all of $\mathbb{R}^S$. Thus R_{fg} is generally a formal construction.

There is always an embedding $\phi: S/R_{fg} \to S/R_f \times S/R_g$ which maps $(s)_{fg} \mapsto ((s)_f, (s)_g)$. Via this embedding a state $s \in S$ is represented by the pair of numbers $(f(s), g(s))$. This embedding ϕ is in general one-to-one but it is onto iff f and g are totally unlinked (to each other).

This product representation can be constructed neatly as a categorical product. Consider the two systems $(S, \{f, 0\})$ and $(S, \{g, 0\})$. The **S**-product of these two systems is $(S \times S, F)$ where $F = \{0, (f, 0), (0, g), (f, g)\} \subset \mathbb{R}^{S \times S}$, with the natural projections. Now consider further the system $(S, \{f, g\})$. There exist **S**-morphisms

$$\phi_1 : (S, \{f, g\}) \to (S, \{f, 0\})$$

and

$$\phi_2 : (S, \{f, g\}) \to (S, \{g, 0\})$$

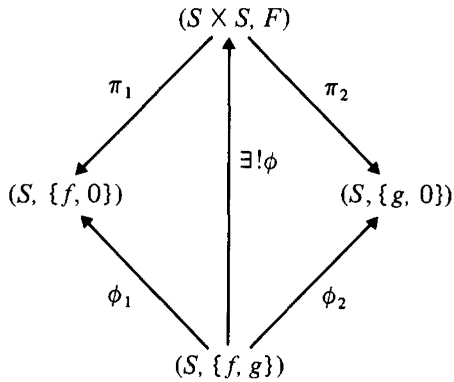
defined by for every $s \in S$

$$\phi_1(s) = s;\ \phi_1 f = f,\ \phi_1 g = 0$$

and

$$\phi_2(s) = s;\ \phi_2 f = 0,\ \phi_2 g = g.$$

So we have the following diagram:



Hence by the universal property of the product there exists a unique ϕ: $(S, \{f, g\}) \to (S \times S, F)$ which makes the diagram commute. Namely, ϕ is defined by sending $s \in S$ to $\phi(s) = (s, s)$, the diagonal map, and by $\phi f = (f, 0)$, $\phi g = (0, g)$. In particular, ϕ being an **S**-morphism implies that ϕ: $S/R_{\{f,g\}} \to S \times S/R_{\{(f,0),(0,g)\}}$. It is clear that ϕ is a one-to-one mapping (on S) and that $R_{\{f,g\}} = R_{fg}$; hence $S/R_{\{f,g\}} = S/R_{fg}$. Also, $S \times S/R_{\{(f,0),(0,g)\}} \cong S/R_f \times S/R_g$. Thus ϕ is indeed the one-to-one map from S/R_{fg} to $S/R_f \times S/R_g$, and that the degree of 'onto-ness' of ϕ is an indication of the lack of linkage between f and g.

7. A Partial Order in **S**. Let (S_1, F_1) and (S_2, F_2) be **S**-objects. Then we define $(S_1, F_1) \leqslant (S_2, F_2)$ if (i) (S_1, F_1) is a monosubobject of (S_2, F_2), i.e. there is a mono ϕ: $(S_1, F_1) \to (S_2, F_2)$; i.e. ϕ: $S_1 \rightarrowtail S_2$ and ϕ: $F_1 \rightarrowtail F_2$ are injective functions; and (ii) S_1 and F_1 are finite sets.

It is clear that $\leqslant$ is reflexive on 'finite **S**-objects', i.e. on (S, F) where S and F are finite sets, and that $\leqslant$ is transitive. Now suppose $(S_1, F_1) \leqslant (S_2, F_2)$ (with mono ϕ) and $(S_2, F_2) \leqslant (S_1, F_1)$ (with mono ψ). Then ϕ: $S_1 \to S_2$ and ϕ: $F_1 \to F_2$ would be injections between finite sets of the same cardinality, and hence are onto. So ϕ: $S_1 \to S_2$ and ϕ: $F_1 \to F_2$ are **Ens**-isomorphisms. Further, the compatibility condition in the definition of **S**-morphisms implies that $S_1/R_f = S_2/R_{\phi f}$ for every $f \in F_1$. Thus (S_1, F_1) is isomorphic to (S_2, F_2) and so $\leqslant$ is antisymmetric (up to isomorphism). Therefore $\leqslant$ is a partial order on (the isomorphism classes of) the finite **S**-objects.

How can we interpret the statement $(S_1, F_1) \leqslant (S_2, F_2)$? First, we have $S_1 \rightarrowtail S_2$ and in fact, for every $s \in S_1$ and every $f \in F_1$ $(s)_f \rightarrowtail (\phi s)_{\phi f}$. So there is a possibility of new states appearing in the whole set and/or in each equivalence class. This reflects *growth* in some respect. The possibility that $F_2 - \phi(F_1)$ is non-empty indicates the *emergence* of more observables as the system becomes 'more advanced'. In particular there is the possibility that $s\ R_{F_1} s'$ in S_1 but there is a $g \in F_2 - \phi(F_1)$ such that $g(\phi s) \neq g(\phi s')$; so states indistinguishable before could be separated—a model of *differentiation*. On the other hand, it could happen that $f(s) \neq f(s')$ in (S_1, F_1) but $(\phi f)(\phi s) = (\phi f)(\phi x')$ in (S_2, F_2)—a model of *integration* or fusion. Also, since in this case 'distinct' states become indistinguishable, there is an indication of *decay*, or 'loss of recognition abilities'.

So it seems that with an appropriate totally ordered subset of this partially ordered set of finite systems, a model of the development–senescence process could be constructed. We shall look at this again later.

Condition (ii) in the definition of $\leqslant$ (that S_1 and F_1 are to be *finite* sets) looks like a very severe mathematical restriction. But in mathematical modelling of natural systems, a finiteness restriction is not unrealistic: all we require is that the sets are finite, and there is no restriction on how small the sets have to be. So the sets could be singletons or have 10^{10} elements or 10^{100} elements or whatever and still be finite. After all, Jeans (1945) defined the universe as a gigantic machine whose future is inexorably fixed by its state at any given moment, that it is "a self-solving system of $6N$ simultaneous differential equations, where N is Eddington's number". Eddington (1939) asserted (perhaps with more poetry than truth) that $N = 2 \times 136 \times 2^{256}$ ($\sim 10^{79}$) is the total number of particles of matter in the universe. The point is that it is a *finite* number. Thus the set of states of a natural system is certainly finite at one time (this is not to be confused with the set of all *possible* states a system can have), and the set of observables on a system at one time is also clearly finite.

8. Dynamics and the Category **D**. A *dynamics* on a set of states S is a one-parameter group (with time $t \in \mathbb{R}$ being the parameter) of bijections on S,

$$T = \{ T_t \in \mathrm{A}(S) \colon t \in \mathbb{R} \}.$$

The domain of the dynamics T is the product set $S \times \mathbb{R}$.

We now turn to the category **D** of dynamical systems. A **D**-object is a pair (S, D), where S is a set (of states, the phase space) and D is a set of dynamics on S.

We always assume that the trivial dynamics I_S, which sends every $x \in S$ to itself for every $t \in \mathbb{R}$, is in D; so, in particular, D is always non-empty. But again for brevity we shall sometimes omit listing I_S in specific examples. Note that if a dynamics is considered to be imposed on S through interactions with the states of other systems, then I_S can be considered as being imposed on S through interaction with the (only) state in S/R_0, where 0 is the 'zero' observable as in Section 4.

A **D**-morphism $\phi \in \mathbf{D}((S_1, D_1), (S_2, D_2))$ is a pair of functions $\phi\colon S_1 \to S_2$ and $\phi\colon D_1 \to D_2$ (with $\phi I_{S_1} = I_{S_2}$), and such that for every $T \in D_1$ and every $t \in \mathbb{R}$, the diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{\ \phi\ } & S_2 \\ {\scriptstyle T_t}\downarrow & & \downarrow{\scriptstyle \phi T_t} \\ S_1 & \xrightarrow[\ \phi\ ]{} & S_2 \end{array}$$

commutes (i.e. for every $x \in S_1$ $\phi(T_t(x)) = \phi T_t(\phi x)$). Composition of **D**-morphisms is defined via compositions of functions component-wise on the states and on the dynamics.

9. Dynamics and Equivalence Relations. Let T be a dynamics on S. T is *compatible* with an equivalence relation R on S if xRy implies $T_t(x)RT_t(y)$ for every t.

If T is compatible with the equivalence relation R on S, then T induces a dynamics T' on the set of 'reduced states' $S' = S/R$; for any equivalence class $(x)_R \in S'$ and $t \in \mathbb{R}$, let $T_t'(x)_R = (T_t(x))_R \in S'$. The dynamics T' is called the *quotient dynamics* on S' induced by T. Note that the equation $T_t'(x)_R = (T_t(x))_R$ states that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \chi\ } & S' \\ {\scriptstyle T_t}\downarrow & & \downarrow{\scriptstyle T_t'} \\ S & \xrightarrow[\ \chi\ ]{} & S' \end{array} \qquad (1)$$

commutes (where $\chi\colon S \to S'$ is the quotient map $x \mapsto (x)_R$). So $\chi\colon (S, \{T\}) \to (S', \{T'\})$, sending each $x \in S$ to $(x)_R \in S'$ and sending T to T', is a **D**-morphism.

There are two complementary questions which can be asked on the connection between dynamics and equivalence relations on S:

(i) Given a dynamics T on S, how can we characterize those equivalence relations R on S with which T is compatible?

(ii) Given an equivalence relation R on S, how can we characterize those dynamics which are compatible with R?

The simpler case when one considers bijections ('automorphisms') $T\colon S \to S$

instead of dynamics was discussed in Rosen (1978) (in which a bijection $T: S \to S$ is defined to be compatible with an equivalence relation R on S if $s\ R\ s'$ implies $Ts\ R\ Ts'$). It was shown that the set of all equivalence relations with which a given bijection $T: S \to S$ is compatible forms a sublattice of the lattice $\mathbf{R}(S)$ of all equivalence relations on S, and that the set of all bijections compatible with a given equivalence relation forms a submonoid of the group $\mathbf{A}(S)$ of all bijections on S; further, the set of all bijections compatible with R, and whose inverses are also compatible with R, forms a subgroup of $\mathbf{A}(S)$. (Note that when both T and T^{-1} are compatible with R, then $s\ R\ s'$ iff $Ts\ R\ Ts'$.)

Let us first expand on these ideas. Define a subset $\mathbf{A}$ of $\mathbf{A}(S)$ to be *compatible* with a family $\mathbf{R}$ of equivalence relations on S if for each $T \in \mathbf{A}$, for each $R \in \mathbf{R}$, and for all $s, s' \in S$, $s\ R\ s'$ implies $Ts\ R\ Ts'$.

We shall now try to answer these two questions:

(i*) Given a subgroup G of $\mathbf{A}(S)$, what are the equivalence relations on S with which G is compatible?

(ii*) Given a sublattice L of $\mathbf{R}(S)$, what are the bijections on S such that they and their inverses are compatible with L?

The same argument used to obtain the results for bijections can easily be generalized to answer these questions. The answer to (i*) is a sublattice of $\mathbf{R}(S)$ which includes the equality relation '=' and the 'universal' relation 'S^2' defined by $s\ S^2\ s'$ iff $s, s' \in S$. Since every bijection is compatible with $=$ and S^2, it is without loss of generality to (and hence we shall) only consider sublattices which contain both of these relations. The answer to (ii*) is a subgroup of $\mathbf{A}(S)$. In fact, this correspondence between the set of all sublattices of $\mathbf{R}(S)$ and the set of all subgroups of $\mathbf{A}(S)$ turns out to be bijective and order (*qua* substructures)-inverting. We can represent this situation by the following self-explanatory diagram:

$$\begin{array}{ccccccc}
\mathbf{R}(S) & \leftarrowtail & L_1 & \leftarrowtail & L_2 & \leftarrowtail & \{=, S\} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
1 & \rightarrowtail & G_1 & \rightarrowtail & G_2 & \rightarrowtail & \mathbf{A}(S)
\end{array}$$

Further, we can define that an equivalence relation R' on S is *conjugate* to another equivalence relation R on S if there is a bijection $T: S \to S$ such that $s\ R'\ s'$ if and only if $Ts\ R\ Ts'$, in which case we shall write $R' = R_T$. Conjugacy is equivalent to the existence of an isomorphism between the collections of equivalence classes S/R and S/R', i.e. there is a bijection between the equivalence classes in S/R and S/R' and corresponding equivalence classes contain the same number of S-elements.

Let L be a sublattice of $\mathbf{R}(S)$. Then we can define a *conjugate* of L to be

$L_T = \{R_T: R \in L\}$ for some $T \in \mathrm{A}(S)$. With these definitions we have the following lemma.

LEMMA. *If (L, G) is a pair of corresponding sublattice-subgroup with respect to compatibility, then the conjugate $(L_T, T^{-1}GT)$ is also a pair of corresponding sublattice-subgroup with respect to compatibility.*

Proof. Let $R \in L$ and $U \in G$. Then for $s, s' \in S$,

$$\begin{aligned} sR_Ts' &\text{ iff } TsRTs' && [\text{definition of } R_T] \\ &\text{ iff } (UTs)R(UTs') && [(R, U) \in (L, G)] \\ &\text{ iff } (TT^{-1}UTs)R(TT^{-1}UTs') && [TT^{-1} = 1_S] \\ &\text{ iff } (T^{-1}UTs)R_T(T^{-1}UTs') && [\text{definition of } R_T] \end{aligned}$$

Thus $T^{-1}\ UT$ and $(T^{-1}\ UT)^{-1}$ are both compatible with R_T. ■

The above results bear a striking resemblance to the Galois theory of field extensions and automorphism groups!

10. Discrete Dynamical Systems. Before going further, let us take a digression in the following direction. Suppose instead of considering 'time' as the continuum of real numbers we consider time as being composed of a succession of 'elementary steps'. Then we can define a *discrete dynamics* on a set S to be a one-parameter group $T = \{T_n: n \in \mathbb{Z}\}$ of bijections from S to S, with the set of integers $\mathbb{Z}$ representing the time parameter. It is clear that for each $n \in \mathbb{Z}$, $T_n = (T_1)^n$. Thus alternatively we can define a discrete dynamics to be the cyclic subgroup of $\mathrm{A}(S)$ generated by a bijection $T = T_1 \in \mathrm{A}(S)$, i.e. a discrete dynamics is $\langle T \rangle = \{T^n : n \in \mathbb{Z}\} \subset \mathrm{A}(S)$. With this terminology, the powers of T in $\langle T \rangle$ become interpretable as *instants of logical time,* and the transition $x \mapsto Tx$, or $T^{n-1}x \mapsto T^n_x$ in general, is an *elementary step* of the dynamics. Further, with appropriate modifications, all the dicussions on the category **D** still go through with $\mathbb{R}$ replaced by $\mathbb{Z}$ and we would have the category of *discrete* dynamical systems instead.

This situation is, of course, closely related to the 'continual' dynamical system as a one-parameter group of bijections $T = \{T_t: t \in \mathbb{R}\}$ on a phase space S. For any real number r we can consider the cyclic subgroup generated by T_r. Then $\langle T \rangle = \{T_r^n = T_{nr}: n \in \mathbb{Z}\}$ defines a discrete dynamics on S. This method of 'discretization' is used, for example, in obtaining numerical solutions of differential equations. Note, however, that this procedure only goes one way: we can obtain a discrete time from a continuous time by choosing the size of an elementary step $t = r$, but starting from a discrete dynamics $\langle T \rangle$, in general we cannot embed it into a continuous one-parameter group of bijections. A further discussion of this aspect can be found in Rosen (1982).

Now let us try to answer questions (i) and (ii) in Section 9 for discrete dynamics.

Given a discrete dynamics $\langle T\rangle = \{T^n: n \in \mathbb{Z}\}$, $T \in \mathbf{A}(S)$, since $\langle T\rangle$ is in particular a (cyclic) subgroup of $\mathbf{A}(S)$, the results of Section 9 tell us that the set of all equivalence relations with which $\langle T\rangle$ is compatible forms a sublattice of $\mathbf{R}(S)$. It is easy to see that $\langle T\rangle$ is compatible with an equivalence relation R on S iff both T and T^{-1} are compatible with R, so we only need to check the compatibility of the generator and its inverse with R.

Since given an equivalence relation R on S, the set of all bijections on S compatible with R and whose inverses are also compatible with R forms a subgroup G of $\mathbf{A}(S)$, the set of all discrete dynamics on S compatible with R is the collection of all cyclic subgroups of G, i.e. the collection of all homomorphic images of $(\mathbb{Z}, +)$ in G.

Finally, we will try to answer, at least partially, the two questions in Section 9 for continual dynamical systems.

Given a dynamics T on S, the collection of bijections $\{T_t: t \in \mathbb{R}\}$ is a subgroup of $\mathbf{A}(S)$. Thus the set of all equivalence relations with which T is compatible is a sublattice of $\mathbf{R}(S)$.

The second question is more difficult. Given an equivalence relation R on S, we obtain a subgroup G of $\mathbf{A}(S)$ of all bijections on S compatible with R. Now we have to find subgroups of G which are (isomorphic to) continuous one-parameter groups indexed by $\mathbb{R}$. Whereas for the discrete dynamics case we can get the set $\{\{T^n: n \in \mathbb{Z}\}: T \in G\}$ quite easily, here there is no trivial way to look for homomorphic images of $(\mathbb{R}, +)$ in G. Note, however, that there is at least one dynamics compatible with R, namely the trivial dynamics $I_S = \{1_S: t \in \mathbb{R}\} \subset G$.

11. Observables and Induced Topologies. Let f be an observable on the set of states S, i.e. $f: S \to \mathbb{R}$. In the construction of the category **S** we could have used equivalence classes of functions in $\mathbb{R}^S/\sim$ as observables (Section 3) because there the only relevant property of f was the equivalence relation R_f it imposed on S. But such a definition of **S**-objects apparently leads to difficulties in some categorical constructions and creates problems in topological considerations.

But we should recall the comments in Section 4 and note that for $f, g \in \mathbb{R}^S$ and $f \sim g$ (i.e. $R_f = R_g$), $(S, \{f\})$ and $(S, \{g\})$ are isomorphic in **S**, and because all constructions in a category are only 'up to isomorphism', these two **S**-objects (and any constructions with one or the other) are 'indistinguishable' in **S**. What properties of $f \in \mathbb{R}^S$ are hidden if we consider $(f)_\sim \in \mathbb{R}^S/\sim$ instead? We see that since $\mathbb{R}$ is a topological space (with the usual topology), $f: S \to \mathbb{R}$ can induce a topology on S, called the *f-topology,* as follows. A subset of S is *f-open* (respectively, *f-closed*) iff it is the inverse image under f of an open (respective, closed) subset of $\mathbb{R}$. The f-topology is the coarsest topology on S which renders f continuous.

For any constant function f (i.e. for $f \in (0)_{\sim} \in \mathbb{R}^S/\sim$) the f-topology is the indiscrete topology $\{\phi, S\}$ on S, and in general if the range of f is a finite set, then any representative of the class $(f)_{\sim}$ induces the same f-topology on S, and so there is a unique '$(f)_{\sim}$-topology'. But if the range of f is infinite, then it is possible for $f(S)$ to have limit points, in which case different class representatives of $(f)_{\sim}$ may induce different topologies; hence one has to consider $f \in \mathbb{R}^S$ and not $(f)_{\sim} \in \mathbb{R}^S/\sim$ in topological considerations.

Further, since $\mathbb{R}$ is a metric space, a 'distance function' on S can be defined using f. Namely for $x, y \in S$, define $d_f(x, y) = |f(x) - f(y)|$. It is clear that $d_f(x, x) = 0$, that $0 \leqslant d_f(x, y) = d_f(y, x) < +\infty$ and that d_f satisfies the triangle inequality. But $d_f(x, y) = 0$ only means $f(x) = f(y)$ and not necessarily $x = y$, so d_f is a *pseudometric* on S and is a metric iff f is injective. Obviously, the f-topology on S is the pseudometric topology generated by d_f and the quotient f-topology on S/R_f is the induced metric topology. Note that if $x \in G \subset S$ where G is f-open and $d_f(x, y) = 0$, then $y \in G$.

Historically, the ideas of limit and continuity appeared very early in mathematics, notably in geometry, and their role has steadily increased with the development of analysis and its applications to the experimental sciences, since these ideas are closely related to those of *experimental determination* and *approximation.* But since most experimental determinaations are *measurements,* i.e. determinations of one or more numbers, it is hardly surprising that the notions of limit and continuity in mathematics were featured at first only in the theory of real numbers and its outgrowths and fields of application. So in a sense topology has its roots in the process of measurement (i.e. observations) and it is interesting to note that we are now using topology as a tool in the study of the fundamentals of measurement and representation of natural systems.

12. Continuity and Compatibility. Let S be a set, T a dynamics on S and $f \in \mathbb{R}^S$ an observable. Define T to be *f-continuous* if T is a continuous dynamics on S with the f-topology, i.e. if $T: S \times \mathbb{R} \rightarrow S$ is continuous. Recall that T is compatible with the equivalence relation R_f (or simply T is *compatible with f*) if for all $t \in \mathbb{R}$ and for all $x, y \in S$, $f(x) = f(y)$ implies $f(T_t(x)) = f(T_t(y))$. The concepts of f-continuity and compatibility are related as follows.

THEOREM. *Let T be a dynamics on S and $f \in \mathbb{R}^S$. If T is f-continuous, then T is compatible with f.*

Proof. If T is f-continuous, then each T_t is continuous on S with the

f-topology. This means that for all $x \in S$ and for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d_f(x, y) = |f(x) - f(y)| < \delta$ implies $d_f(T_t(x), T_t(y)) = |f(T_t(x)) - f(T_t(y))| < \epsilon$. Now let x and $y \in S$ be such that $f(x) = f(y)$. Then for every $\epsilon > 0$, $d_f(x, y) = |f(x) - f(y)| = 0 < \delta(\epsilon)$. Hence $d_f(T_t(x), T_t(y)) = |f(T_t(x)) - f(T_t(y))| < \epsilon$; so $|f(T_t(x)) - f(T_t(y))| = 0$, whence $f(T_t(x)) = f(T_t(y))$. Thus T is compatible with f. ■

It is clear that the converse of this theorem is not true.

13. The Category of Natural Systems. The categories **S** and **D**, representing the static and dynamic aspects of natural systems, are now amalgamated into a category denoted by **N**. Explicitly, an **N**-object, a natural system, is a triple (S, F, D), where (S, F) is an **S**-object and (S, D) is a **D**-object.

We shall also consider S as a set on which different topologies can be defined. In particular we shall consider the topological spaces (S, τ), where τ can be the f-topology for any $f \in F$.

$\phi \in \mathbf{N}((S_1, F_1, D_1), (S_2, F_2, D_2))$, i.e. ϕ is an **N**-morphism, if $\phi \in \mathbf{S}((S_1, F_1), (S_2, F_2))$ and $\phi \in \mathbf{D}((S_1, D_1), (S_2, D_2))$. Thus ϕ is a mapping of the sets $S_1 \to S_2$, $F_1 \to F_2$ and $D_1 \to D_2$ such that on (S, F) it satisfies the conditions of Section 4 and on (S, D) it satisfies the conditions of Section 8.

We do not impose any relations for ϕ on (F, D). So, for example, $T \in D_1$ may be f-continuous for some $f \in F_1$, but $\phi T \in D_2$ is not required to be ϕf-continuous; and we can have T compatible with f but ϕT not compatible with ϕf, and so on.

The identity **N**-morphism $1_{(S,F,D)}$ is clearly the amalgamation of $1_{(S,F)}$ and $1_{(S,D)}$, i.e. $1_{(S,F,D)}$ sends each $x \mapsto x \in S$, $f \mapsto f \in F$ and $T \mapsto T \in D$.

Composition of **N**-morphisms is defined 'component-wise' and is clearly associative with identity $1_{(S,F,D)}$.

If ϕ: $(S_1, F_1, D_1) \to (S_2, F_2, D_2)$ is an **N**-isomorphism, then (S_1, F_1) and (S_2, F_2) are **S**-isomorphic and (S_1, D_1) and (S_2, D_2) are **D**-isomorphic. Note that even between **N**-isomorphic systems the continuity and compatibility properties of the dynamics and observables are not necessarily preserved. This is due to the fact that the observables are less 'well-behaved' and that $\sim$-equivalent observables, which are not necessarily topologically equivalent, are **S**-isomorphic. This apparent shortcoming, contrariwise, turns out to be of great interest; some of these 'bifurcation phenomena' will be discussed next.

14. Bifurcations. The **N**-object (S, F, D) contains many different mathematical descriptions of the same system. There are many interesting questions about their connections one can ask: How do the descriptions of S obtained from one set of observables $\{f_1, f_2, \ldots, f_n\}$ compare with those from another set $\{g_1, g_2, \ldots, g_m\}$? How can one combine these descriptions to obtain a more comprehensive picture of S? To what extent does a knowledge that two states s_1 and s_2 appear 'close' under the pseudometric on S induced by $\{f_1, f_2, \ldots, f_n\}$ imply that these same states appear close with respect to $\{g_1, g_2, \ldots, g_m\}$? How does a dynamics T appear when viewed through an observable f? And conversely, how does an observable appear after the passage of dynamical process?

These questions are intimately related to the notions of *stability* and *bifurcation* (Thom, 1975) and devolve once again to the notion of linkage (Section 6), and thence back to the essence of the modelling relation itself.

In Section 6 we considered those relationships among observables $f, g, \ldots$ based on the concept of linkage, which depends only on the equivalence classes of the relations $R_f, R_g, \ldots$. Lack of linkage leads to 'state transitions' $s \to s'$, which are undetectable by f (i.e. $f(s) = f(s')$) but are detected by g (i.e. $g(s) \neq g(s')$).

Next we shall try to answer the question: If s is 'close' to s' under (the pseudometric f-topology induced by) f, when will s also be close to s' under g? Thus we are considering the extent to which a state transition $s \to s'$ which is 'almost' undetectable by f (i.e. $|f(s) - f(s')|$ is small) is likewise almost undetectable by g (i.e. $|g(s) - g(s')|$ is small).

Definition. Let f, g be observables on S and d_f, d_g be the corresponding induced pseudometrics (Section 11). The state $s \in S$ is a *stable point of* g *with respect to* f if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for $s' \in S$, $d_f(s, s') = |f(s) - f(s')| < \delta$ implies $d_g(s, s') = |g(s) - g(s')| < \epsilon$.

The definition is equivalent to each of the following:

(i) the identity map of the set S from (S, d_f) to (S, d_g) is continuous at s;
(ii) the f-open neighbourhood system of s refines the g-open neighbourhood system of s;
(iii) (roughly) every state 'f-close' around s is also 'g-close' around s.

The following results are clear:

(a) If $s \in S$ is a stable point of g with respect to f then $(s)_f \subset (s)_g$.
(b) If $s \in S$ is a stable point of g with respect to f and $s' \in (s)_f$, then s' is also a stable point of g with respect to f.
(c) The set of stable points of g with respect to f is an f-open subset of S.

Definition. The complement of the set of stable points of g with respect to f is the *bifurcation set* of g with respect to f.

Intuitively, near a bifurcation point of g with respect to f, the proximity of two states s and s' as viewed by the observable f does not imply their proximity as viewed by g. In other words, the bifurcation set is the set of states at which the g-description does not agree with the f-description in their metrical aspects, i.e. the two descriptions convey essentially different information.

It follows from (c) above that the bifurcation set of g with respect to f is an f-closed subset of S. Also, it is clear that the bifurcation set of g with respect to f is empty iff the f-topology is finer than the g-topology on S, in which case R_f refines R_g.

In the previous discussion we can interchange the role of f and g and obtain the opposite concept of stable and bifurcation points of f with respect to g. These are, in general, quite different from those obtained from g with respect to f. Thus, given a pair of observables, we obtain two distinct notions of stability and bifurcation, depending on which description is chosen as the reference.

Let us consider the case when f and g are two observables on S such that the bifurcation sets of f with respect to g and of g with respect to f are both empty. Then 1_S: $(S, d_f) \to (S, d_g)$ is a homeomorphism and d_f and d_g are equivalent pseudometrics, i.e. f and g induce the same topology on S. Under these circumstances it is appropriate to say that f and g are *topologically equivalent* (as opposed to *algebraically equivalent*: $f \sim g$ when $R_f = R_g$). Note that by result (a) above, topological equivalence implies algebraic equivalence; but not conversely. Stated another way, we have the following theorem.

THEOREM. *If two observables induce the same topology on the set of states, then they are totally linked to each other.*

15. Incompatibility. We say in Section 9 that when a dynamics T on a set S is compatible with an equivalence relation R on S, T induces a quotient dynamics on the set of reduced states S/R. In this section we shall investigate what happens when a dynamics T is *not* compatible with an equivalence relation R in the special case when $R = R_F$, where F is a family of observables on S.

If T is not compatible with R_F, then there are states $s, s' \in S$ for which $s \; R_F \; s'$, but for some $t \in \mathbb{R}$ $T_t s$ and $T_t s'$ are not R_f-related. That is, T splits equivalence classes of R_F. Putting it another way, the two indistinguishable states s and s' (under F) have now 'differentiated' through the action of the dynamics T, and this differentiation is visible using the observables in F. (The term 'differentiation' is used deliberately here to suggest the connection of this with biological differentiation. See Section 19.)

From the viewpoint of an observer equipped with meters for observables in F, the states s and s' appeared to be the same. But through the course of the dynamics T the observer detects two different states $T_t s$ and $T_t s'$. It would appear that the same initial state under the same conditions has given rise to two distinct states, a contradiction to causality. The problem here is, of course, that one usually assumes that one has a complete set of observables F for the description of S (i.e. $S/R_F = S$, or R_F is the equality relation). The standard way out is to pull in statistics and to observe many copies of $(s)_{R_F}$ under the passage of T. The relative frequencies of the resulting states $(T_t s')_{R_F}$ (where $s' \in (s)_{R_F}$) are then associated with *transition probabilities* from s to $T_t s'$. In other words, the incompatibility of T with R is usually interpreted in stochastic terms.

One can (rather boldly) make the suggestion that all processes occuring in nature are deterministic and that the apparent stochasticity is simply a consequence of employing an incomplete description as if it were complete. So one does not need statistical tools *if* one has a complete description of the system. But that is a rather big 'if' because we are limited in our means of observation, measurement and understanding, and to obtain a complete description of every natural system is really to find the philosopher's stone! Thus statistics plays a role in science as a matter of necessity. Further discussions of the interplay between causality and chance can be found in Bohm (1957) and also in Belinfante (1973) on the theory of 'hidden variables'.

16. Stability and Commutativity. We just considered the situation when a 'change of state' $s \to s'$ which is undetectable by an observable f becomes detectable by the observable $g = f \circ T_t$. This, of course, is again intimately related to the concept of linkage. After all, given an observable f and a dynamics T on S, for each $t \in \mathbb{R}$ $f \circ T_t$ is an observable on S. To say that F is compatible with R_f (i.e. f) simply means that R_f refines each $R_{f \circ T_t}$, or that each $f \circ T_t$ is totally linked to f.

With this in mind, the next natural question to ask is: If s is close to s' under f, when will s also be close to s' under f after the action of a dynamics T? This problem can then clearly be studied by reducing to consider stable and bifurcation points of $f \circ T_t$ with respect to f. And the result from the previous sections can be appropriately modified and used.

Since the compatibility of T with f is equivalent to the commutativity of the diagram (1) of Section 9, the study of stability and bifurcation can be formulated in the category **N** as the 'approximate commutativity' of this diagram.

Finally, there is an interesting possibility that a state $s \in S$ can be a stable point of $f \circ T_t$ with respect to f for all t less than a 'critical time' t_c, and then for $t > t_c$ s becomes a bifurcation point of $f \circ T_t$ with respect to f. In other words, at $t = t_c$ we have a *catastrophe* [in the sense of Thom (1975)]. Alternatively we can consider $t \in \mathbb{R}$ as an 'order parameter' and at $t = t_c$ we have a 'change of scheme' from an old structure to a new structure through an instability. This area is a further topic of investigation and we shall not deal with it here. A good reference set is the Springer

Series in Synergetics, especially the introductory Vol. 1 (Haken, 1978) and Vol. 4 (Güttinger and Eikemeier, 1979) on Structural Stability in Physics.

17. Cellular Dynamics. We come now to the final subject of this paper. We want to show how the mathematical formalism of categorical system theory we developed may be applied to various biological topics. Let us consider an example of a natural system—a set equipped with a collection of observables and a collection of dynamics. What is more natural, in biology, then to begin with a cell? Differences among cells can be referred to differences in their molecular constitution ultimately specified by their genome, and hence to differences in the relative concentrations of constituent molecules. Thus it is reasonable to suppose that the state variables in terms of which we describe a cell at a particular instant of time shall be concentrations of chemical substances. These are observables of the cell.

Chemical substances interact with one another and their concentration will be changing in time. The 'cellular dynamics' are given by the set of differential equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(x_1, \ldots, x_n), \tag{1}$$

where the x_is denote the concentrations of the chemicals in the cell and the functions f_i are determined by the specific reactions. The stability and bifurcation properties of such a dynamical system (1) may then be directly interpreted in terms of cellular behaviour. Further, the functions f_i, which specify the rate of change of each of the state variables, are themselves observables of the cell.

There are plenty of other examples in biology of sets equipped with observables and dynamics: one only has to go up the hierarchical 'ladder' and consider tissues, organs, organisms, ecosystems and so on. And it is this abundance of natural systems that leads to the use of categories as the mathematical tool of organization.

Category theory also provides a natural framework in which to analyse the important concept of *subsystem*, via the categorical definition of *subobject*. For example, a subgroup of a group is not simply any subset: it has to satisfy the group axioms, i.e. it has to be a group itself. Similarly, not every subset of a natural system can be a subsystem: it has to be a system itself. In particular, the dynamics restricted to the subset have to remain as dynamics; equivalently, the subset has to be functionally isolable. This is, for instance, the situation involved in the concept of the *active site* of an enzyme. The active site is dynamically linked to the rest of the enzyme (one theory being that it is simply a local maximum of the energetics) and cannot be separated by any physical procedure. Fractionation destroys the enzyme dynamics. In other words, the concept of an active site only makes sense in the larger domain of an enzyme, the former not being a natural system itself.

18. A Partial Order in **N**. Let (S_1, F_1, D_1) and (S_2, F_2, D_2) be **N**-objects. Then define $(S_1, F_1, D_1) \leqslant (S_2, F_2, D_2)$ if

(i) there is an **N**-mono $\phi: (S_1, F_1, D_1) \to (S_2, F_2, D_2)$; i.e. $\phi: S_1 \rightarrowtail S_2$, $F_1 \rightarrowtail F_2$ and $D_1 \rightarrowtail D_2$ are injective functions; and
(ii) S_1, F_1 and D_1 are finite sets.

Note first that if $(S_1, F_1, D_1) \leqslant (S_2, F_2, D_2)$ then $(S_1, F_1) \leqslant (S_2, F_2)$ in **S** (Section 7). Also, since S_1 is a finite set, for each $f \in F_1$ $f(S_1)$ is a finite subset of $\mathbb{R}$, and hence any representative of the class $(f)_\sim$ induces the same f-topology on S_1. In other words there is a unique $(f)_\sim$-topology on S_1 (Section 11).

It is clear that $\leqslant$ on the finite **N**-objects [i.e. on (S, F, D), where S, F and D are finite sets] is reflexive and that $\leqslant$ is transitive. It is also clear, using a similar argument as in Section 7, that if $(S_1, F_1, D_1) \leqslant (S_2, F_2, D_2)$ and $(S_2, F_2, D_2) \leqslant (S_1, F_1, D_1)$ then the **N**-monos involved are in fact **N**-isomorphisms. So $\leqslant$ is antisymmetric (up to isomorphism, as usual). Therefore $\leqslant$ is a partial order on (the isomorphism classes of) the finite **N**-objects.

19. Growth and Aging as a Partial Order. What does this partial order $\leqslant$ in **N** have to do with development and senescence? Since $(S_1, F_1, D_1) \leqslant (S_2, F_2, D_2)$ in **N** implies in particular that $(S_1, F_1) \leqslant (S_2, F_2)$ in **S**, we can include the discussions from Section 7. Let us see what we have.

With an **N**-mono $\phi: (S_1, F_1, D_1) \leqslant (S_2, F_2, D_2)$, the possibility that ϕ is not onto models *growth*. If $S_2 - \phi(S_1)$ is not empty, then more elements (states) have appeared in the second system, an increase in *size*. If $F_2 - \phi(F_1)$ is non-empty, then there are more observables in the second system, an increase in *complexity*. If $D_2 - \phi(D_1)$ is non-empty, then more dynamics can be imposed on the second system, an increase in *interactive ability*. Collectively, the appearance of these new modes of structures, organization and behaviour falls into the description of the sometimes-puzzling biodynamical phenomenon termed *emergence*.

When $g \in F_2 - \phi(F_1)$ and $s, s' \in S_1$ are such that $s\, R_{F_1}\, s'$ but $g(\phi s) \neq g(\phi s')$, indistinguishable states in S_1 are now separated in S_2 because of an increase in complexity, an alternate description. When $T \in D_1$ is compatible with R_{F_1} but ϕT is not compatible with some $g \in F_2$ (Section 15), we have the case that indistinguishable states in S_1 become separated in S_2 because of an interaction through an additional dynamics. Both of these cases indicate the presence of *differentiation* going from (S_1, F_1, D_1) to (S_2, F_2, D_2).

Supposing now $s, s' \in S_1$ and $f \in F_1$ are such that $f(s) \neq f(s')$ yet $(\phi f)(\phi s) = (\phi f)(\phi s')$, then distinct states become 'the same'. Also, when a dynamics is not compatible with an equivalence relation different equivalence classes may appear to 'fuse' together in the course of the dynamical process. These serve as models for biological *integration* when interpreted 'positively' and for *decay* ('loss of information') when interpreted 'negatively'.

When a change of continuity properties occurs going from (S_1, F_1, D_1) to (S_2, F_2, D_2), e.g. when T is f-continuous ($T \in D_1$, $f \in F_1$) but ϕT is not ϕf-continuous, or when the linkage between ϕf and ϕg is different from that between f and g ($f, g \in F_1$) etc., it could be interpreted as change of biological structures and functions. This kind of apparently discontinuous change in biological systems again falls into the area of 'emergence'. The generation of emergent novelties is highly characteristic of biological systems, and in our formalism of categorial system theory it is a natural consequence of the definition of $\leqslant$.

With all of the above in mind we can now make a formal definition of aging and growth.

Definition. Let (S_1, F_1, D_1) and (S_2, F_2, D_2) be finite natural systems. Then (S_1, F_1, D_1) is *younger* than (S_2, F_2, D_2) (and the latter is *older* than the former) if $(S_1, F_1, D_1) \leqslant (S_2, F_2, D_2)$. In other words *aging* is defined as the partial order $\leqslant$ on the finite natural systems.

The more we know about aging, the more irreversible a process it seems to be. That is why ideas from irreversible thermodynamics and dissipative systems are used to model aging [see, for example, Richardson (1980)]. The irreversibility of aging is captured in the above definition. If $(S_1, F_1, D_1) \leqslant (S_2, F_2, D_2)$, then, of course, in general we do not have $(S_2, F_2, D_2) \leqslant (S_1, F_1, D_1)$. Indeed, if the latter holds we would have $(S_1, F_1, D_2) \cong (S_2, F_2, D_2)$ in **N**, in which case the two systems are of the 'same age'. So the partial order $\leqslant$ gives a unidirectionality of aging and growth.

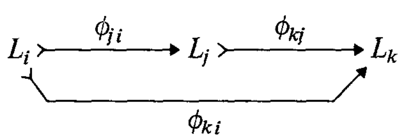
Note, however, that it is possible that there is a subsystem (S, F, D) of $(\phi(S_1), \phi(F_1), \phi(D_1)) \subset (S_2, F_2, D_2)$ on which ϕ^{-1} exists and is an **N**-mono. In other words, aging results from properties and relations of whole systems, and it does not forbid the possibility that one or more of the component subsystems gives opposite contributions. Aging is a *collective* (*cooperative*) *phenomenon* of many processes, some of which may appear to defy aging.

20. The Concept of the Organism. When one thinks of collective phenomena in which the discrete constitutive individuals are modified in their behaviour through interactions among one another to fit into the pattern of a larger collective set, and the whole is more than and different from a simple addition of its parts, living organisms would seem to be the ideal example.

A view of the organism is from that of organic behaviour, in the context of a relational biology, as conceived originally by Rashevsky (1960). On the integrative aspects of behaviour, it was Rashevsky's idea that the organisms are recognized as such because we can observe homologies in their behaviours, regardless of the physical structures through which these observations are made. Thus all organisms manifest the same set of basic and ubiquitous biological functions, and through this manifestation organisms can be mapped on one another in such a way as to preserve these basic relations. This idea led to the formulation of Rashevsky's principle of biotopological mappings and Rosen's (categorical) (M, R)-systems (Rosen, 1972). On the adaptive and predictive character of organic behaviour, one is led to the classical (optimal) control theory (Rosen 1980) and Rosen's theory of anticipatory systems (Rosen, 1982).

Along these lines of relational biology, the following is a description of the developmental processes of an organism from a categorical standpoint.

Definition. Let **L** consist of: (1) a collection $\{L_i = (S_i, F_i, D_i): i \in I = [0, 1]\}$ of **N**-objects such that for $i \leqslant j$ in I, $L_i \leqslant L_j$ in **N** (so in particular the L_is are 'finite' **N**-objects); and (2) for each pair $i, j \in I$ with $i \leqslant j$, a hom set $\mathbf{L}(L_i, L_j)$ containing a *single* **N**-mono $\phi_{ji}: L_i \to L_j$, such that for $i \leqslant j \leqslant k$ in I, $\phi_{kj} \circ \phi_{ji} = \phi_{ki}$



and if $i > j$ in I then there is no **N**-morphism in **L** from L_i to L_j: i.e. $\mathbf{L}(L_i, L_j) = \emptyset$ for $i > j$.

It follows from (2) that for each $i \in I$, $\{\phi_{ii} = 1_{L_i}\} = \mathbf{L}(L_i, L_i)$. Thus **L** is a subcategory of **N** and the set of **L**-objects is totally ordered by $\leqslant$.

Definition. An *organism* is a natural system which is (a realization of) **L** for an appropriately chosen family $\{L_i\}$ of **N**-objects and an appropriately chosen collection $\{\phi_{ji}\}$ of **N**-monos satisfying the definition of **L**.

Considering the results we have from the discussions on aging, it is quite reasonable to make the above definition of an organism from the standpoint of categorical system theory. The totally ordered set $(I, \leqslant)$ is an index of *age* and the order $\leqslant$ on **L** is the process of aging. The instant $i = 0$ can be considered as the moment of *conception* of an organism (when *life* begins) and the instant $i = 1$ is *death*. The developmental processes of the organism are reflected in the systems (S_i, F_i, D_i), the morphisms ϕ_{ji}, and in particular in the evolution of the systems as the index $i \in I$ increases from 0 to 1.

It is an appropriate place here to mention a new subject of study started by Rosen (1982): that of *anticipatory systems*. The basis for this theory is the recognition that most (if not all) of the biological behaviour is of an anticipatory rather than a reactive nature. Let us see if our definition of an organism has incorporated into it this anticipatory character. From the total order on the set of **L**-objects we can conclude that certain things *cannot* happen as the organism develops (i.e. as i increases). For example, it cannot happen that $f(s) = f(s')$ but $(\phi_{ji}f)(\phi_{ji}s) \neq (\phi_{ji}f)(\phi_{ji}s')$, it cannot happen that $T \neq T'$ but $\phi_{ji}T = \phi_{ji}T'$ and so on. So this list of impossible happenings can be considered as a prediction of things to come (or, rather, *not* to come). Also, the linkage imposed on the states by the dynamics coupled with causality determines how a present state can be regarded as a model of a future state. There are many qualities like these one can list. A formal study of 'the category of anticipatory systems' is a perspective for the future.

21. Organismic Sets. Organismic sets were built by Nicolas Rashevsky as a representation of biological organisms and societies on a relational basis (Rashevsky, 1972), and a wide range of biological and social phenomena were explained within this framework.

The idea was first started by the observation (!) of the remarkable relational similarities among physics, biology and sociology. The phenomena are properties of collections of things that are capable of performing certain activities which result in certain products. This led to the suggestion of the existence of a conceptual superstructure of which physics, biology and sociology are three parallel branches, each partially isomorphic to the other two. This conceptual superstructure is an *organismic set*.

Even before we formally define what an organismic set is, we can see that the setting is perfect for a category theory to be postulated. After all, we are looking at a class of mathematical objects with the same structure.

Definition. An *organismic set* is a finite set S such that:

(1) Corresponding to each element $e_i \in S$ there is a set S_i^a of *activities* which e_i is capable of performing, and there is a set S_i^p of *products* which e_i can make.

(2) The set of all potential activities $S^a = \bigcup_i S_i^a$ and the set of all products $S^p = \bigcup_i S_i^p$ of the organismic set both have cardinality greater than one.

(3) In a given environment E at a given time t, each e_i only exhibits a proper subset $S_i^a(E, t)$ of S_i^a and makes only a proper subset $S_i^p(E, t)$ of S_i^p. This models *adaptation* and *development* as E and t vary.

(4) S is partitioned into three disjoint subsets S_1, S_2 and S_3 such that $S - S_3 = S_1 \cup S_2$ is a 'normal' organismic set in itself (i.e. S_3 and its associated S_3^a and S_3^p are the apparently 'useless' parts of the organismic

set S), $S - S_2$ is an organismic set which can exist but will not develop and S_1 is the *core* of the organismic set so that $S - S_1$ cannot exist (i.e. S_1^a and S_1^p are necessary and sufficient for at least a short-range existence of S).

(5) Taken alone, i.e. removed from S, each $e_i \in S$ has a *survival time* t_i during which they can exist without the availability of S^p. t_i is short compared to the life span of S.

(6) Elements in S^p (i.e. products) act on various e_is and so induce a non-empty set of relations S_R within S. It is these relations that make us recognize an organism or a society as such. Members of S_R are in general k-ary relations with $k \geqslant 2$.

The above is a very much simplified version of Rashevsky's original formal definition of organismic sets. Attempts to study organismic sets using category theory were made [see Baianu (1980) for a list of references], but ordinary categories did not seem to be sufficient and the theory of 'super-categories' was developed. For our purpose we shall simply consider the pieces we took and see if we can fit them into the formalism of our category **N**.

22. Organismic Category. Let **O** be the category of organismic sets; i.e. an **O**-object is an organismic set S. The elements $e_i \in S$ can be considered as 'states' of the system S. Members of the sets S^a and S^p can be considered as 'observables' on S. In particular the relations in S_R can be interpreted as the observable-induced relations. Although the real-valued requirement of the observables is not met in this case, one can always 'digitize' S^a and S^p and impose artificial numbers on them (as in Gödel numbers). Of course, this digitization has to be done in such a way that there is minimal loss of information. This situation is perhaps similar to that of numerical taxonomy when we give numerical values to the different taxa. The activities and products in S^a and S^p can also be considered as dynamics induced on the system. Even the survival time t_i of each e_i can be interpreted as the 'inherent dynamics' of the element. Thus the map $S \mapsto (S, F = S^a \cup S^p, D = S^a \cup S^p \cup \{t_i\})$ is one on the objects of a functor $\mathbf{O} \to \mathbf{N}$. The various other properties of S can be looked upon as further structures on the objects; in other words, **O** can be studied as a category of **N**-objects with structure.

What, then, are the admissible **N**-morphisms ϕ between organismic sets S and S'? Besides the usual requirements of being an **N**-morphism (which implies ϕ: $S \to S'$, $S_i^a \to S_i'^a$, $S_i^p \to S_i'^p$, and $\{t_i\} \to \{t_i'\}$), we would like it to preserve the other structures as well. Thus we want $\phi(S_i^a(E, t)) \subset S_i'^a(E, t)$, $\phi(S_i^p(E, t)) \subset S_i'^p(E, t)$ and ϕ: $S_1 \to S_1'$, $S_2 \to S_2'$, $S_3 \to S_3'$ etc. With such a definition **O**-isomorphic objects would then be abstractly identical organismic sets. Admittedly the above contains many hand-waving arguments. But the **O** → **N** association looks rather promising, and the **O**-morphisms do indeed look like the transformation between abstract diagrams representing biological systems, i.e. the biotopological mappings between graphs of organisms (cf. Rashevsky, 1960). The above should be able to acquire mathematical rigour upon further 'hard' analysis.

23. Epilogue. Let me end this paper by pointing out that it represents part of a larger field of studies of relational biology started by Rashevsky. It was his insight to recognize that our direct interest in biological systems is primarily function and behaviour, not structure. Rosen studied general cellular organizations in this context in his metabolism-repair systems (Rosen, 1958). The natural systems studied in this monograph share a common feature with the (M, R)-systems in that they are characterized completely in functional and organizational terms, entirely devoid of physical structure. Moreover, the base sets (state spaces) have no *a priori* algebraic structure and hence can be particularized to various specific examples when they are further equipped with linearity, topologies and so forth. The treatment is of a formal character and interesting consequences include the many possible physical and biological interpretations (i.e. realizations) of the formalism, some of which are discussed in the last few sections. Other will follow.

> "Any particular or isolated biological phenomenon or group of phenomena admits of necessity an explanation in terms of a mathematical model" (Rashevsky).

I thank Professor Robert Rosen for his inspiration.

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