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A PHENOMENOLOGICAL CALCULUS FOR ANISOTROPIC SYSTEMS

ABSTRACT. The phenomenological calculus is a relational paradigm for complex systems, closely related in substance and spirit to Robert Rosen's own approach. Its mathematical language is multilinear algebra. The epistemological exploration continues in this paper, with the expansion of the phenomenological calculus into the realm of anisotropy.

KEY WORDS: anisotropic systems, phenomenological calculus, triadic response tensors

1. PROLEGOMENON

Once upon a time, in Halifax, Nova Scotia, there was a Red House, that housed the Biomathematics Program of Dalhousie University. For a precious but unfortunately brief time (1975–1985), it was one of the most innovative and fruitful programs for research and teaching in theoretical biology in the world. The program's creation and Robert Rosen's association with it as Killam Professor of Biomathematics were mainly due to the efforts of I.W Richardson. Rosen came to Dalhousie in 1975, and stayed until his retirement in 1993.

A specimen of the output of the Red House program, symbol of its material realization, is the posthumously (with respect to the Red House) published volume *Theoretical Biology and Complexity: Three Essays on the Natural Philosophy of Complex Systems* (Rosen ed., 1985). It was written by the "Red House triumvirate" that were involved with the science of relational biology: Rosen, Richardson, and Louie. The three essays are 'The dynamics and energetics of complex real systems' by I. W. Richardson, 'Categorical system theory' by A. H. Louie, and 'Organisms as causal systems which are not mechanisms: an essay into the nature of complexity' by Robert Rosen. Although separately conceived and written, the three essays are closely related in substance and spirit. Rosen wrote in the Preface of the volume:

"Let us turn first to the scientific threads that relate the three contributions. Although very different in emphasis and in thrust, they all spring from a common concern: to grasp and comprehend the material basis of living systems. I believe that we each began with a conviction that contemporary physics already contained the necessary universals with which to cope with the phenomena of life and that therefore only a clever rearrangement and redeployment of these universals would suffice to bring them to bear effectively on biology. I believe that we each came separately, and with great reluctance, to admit the possibility that this conviction might not be true, and hence that a true theory of the organism required new physics and new epistemology. And again separately, we realized that the measurement process, which lies at the very heart of every mode of system description, provides perhaps the only safe and fundamental point of departure for building a comprehensive theory, not only of organisms, but of natural systems in general. This premise is the primary thread that binds the essays in this volume together. As Dr. Richardson likes to say, such an approach restores to our fragmented sciences the kind of integration and unity they possessed in an earlier time, when scientists regarded themselves as natural philosophers."

An organism is a complex system. One characterization of a complex system is that it admits *alternate descriptions* (Rosen, 1976). In Section 5.4 of Rosen (1978), one finds this

"...a [natural] system is *simple* to the extent that a single description suffices to account for our interaction of the system; it is *complex* to the extent that this fails to be true."

Next, in Section 5.7 of Rosen (1985), one finds this:

"*...we are going to define a system to be complex to the extent that we can observe it in non-equivalent ways.*"

Rosen's final refinement of the definitions of simple and complex systems appears in Chapter 19 of Rosen (2000):

"A system is *simple* if all of its models are simulable. A system that is not simple, and that accordingly must have a nonsimulable model, is *complex*."

While the connections of the three approaches to natural philosophy in Rosen (ed. 1985) (other than the common thread of measurement) may not be immediately apparent, they at the very least provide, in the same spirit, alternate descriptions of complex natural systems.

The Rosen–Richardson–Louie interconnections are, in fact, much more profound. The phenomenological calculus, the subject of the current paper, is a direct descendent of the Richardson tierce on complex real systems and the Richardson and Rosen (1979) paper on aging and the metrics or time. The Louie tierce is his Rosen-supervised doctoral dissertation, covering topics suggested in Rosen (1978). Its Section VII.D is on the connections between the phenomenological calculus and Louie's categorical system theory, hence by extension between the phenomenological calculus and Rosen's fundamentals of measurement. These connections are discussed in more detail in one of the phenomenological calculus papers, Louie (1983). The Rosen tierce, in which his Aristotelian causality discussion first appears, is an antecedent of his masterwork *Life Itself* (Rosen, 1991).

Rosen's (M,R)-systems are his quintessential models of organisms, whence life. (See the separate Louie paper (Louie, 2005) in this issue of *Axiomathes* for details.) Their representation in the realm of the phenomenological calculus are in Louie and Richardson (1983) and Louie (1983). Another Rosen model, his Stieltjes integral of enzyme-substrate recognition (Edelstein and Rosen, 1978), is discussed in Louie et al. (1982), Richardson and Louie (1983), Louie (1983), and Louie and Richardson (1986). It is quite remarkable that the apparently different methods (efficient causes) in the natural philosophy from the Red House are actually very close in essence (final cause).

From whatever side we approach our principle, we reach the same conclusion.

2. PRAELUDIUM

The phenomenological calculus is a mathematical tool for the representation of complex systems. It has proven to be extremely versatile in its applicability to various biological, physical, and chemical topics: the itinerary includes (in the chronological order of the 10 published papers) dissipation (Richardson, 1980), aging (Richardson et al., 1982), enzyme-substrate recognition (Louie et al., 1982), (M,R)-systems (Louie and Richardson, 1983), chemical dynamics (Richardson and Louie, 1983), protein modeling (Louie, 1983), quantum mechanics (Richardson and Louie, 1986), relativity (Louie and Richardson, 1986), the Gibbs paradox (Richardson, 1989), and membrane transport (Richardson and Louie, 1992).

All the systems considered thus far have been isotropic systems. *Isotropy* is the property of being independent of direction, whence cause and effect are collinear. *Anisotropy*, the opposite of isotropy, is therefore by definition the property of being directionally dependent. Anisotropic systems are ubiquitous, in constrained transport, crystallography, fluid dynamics, thermics, seismology, cosmology: indeed whenever a wave propagates through a medium. In its most general form, an anisotropic system is where cause and effect are not collinear. It may even be argued that nature is anisotropic. Isotropic systems are simply ''weakly anisotropic'' ones, for which linear approximations suffice. We now continue our epistemological journey into the domain of *anisotropic* systems, with a more complex cause-and-effect phenomenology.

3. CARTESIAN RESPONSE TENSORS

The mathematical object of the phenomenological calculus is the *description space* D, a subspace of the space $T_1^1(H)$ of type-(1,1) tensors over a real Hilbert space H. Members of D are dyadics called *response tensors*. In Appendix A, we review enough concepts to make the present paper (more-or-less) self-contained, and give a concise summary of the work that has been done. The reader is invited to refer to our previously published sequence of papers for details.

While the general Hilbert-space setting has many interesting consequences (especially when $H = L^2$, the space of all square-integrable functions) in our phenomenological calculus, we have demonstrated that a good deal of science is already entailed in the original finite-dimensional Euclidean-space setting when $H = \mathbb{R}^n$. In the present paper we shall specialize further, and restrict the type of coordinates to be Cartesian (i.e. orthonormal) in the Euclidean space $H = \mathbb{R}^n$.

This simplification has many useful benefits. There is no need to make a distinction between covariant and contravariant Cartesian tensors, since Cartesian bases are self-dual. The notation, therefore, only requires subscripts for both components and bases. The summation convention is hence repeated (Roman) subscripts imply summation. The metric is the Kronecker delta δ_{ij} in all Cartesian coordinate systems. With Cartesian tensors, we may also relax the restriction of our response tensors as type-(1,1). All that is required is that it is a second-order tensor, since we have $T_0^2(\mathbb{R}^n) = T_1^1(\mathbb{R}^n) = T_2^0(\mathbb{R}^n)$. In Appendix B we give a brief introduction to the multilinear algebra of Cartesian tensors. Now let us present an alternate construction of the description space.

Fix vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$, and allow vectors $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_m \in \mathbb{R}^n$ to vary. A *response tensor* is the dyadic

$$\mathsf{R} = \mathbf{a}_i\mathbf{F}_i \tag{1}$$

(sum over i from 1 to m). The set

$$D = \{\mathsf{R} = \mathbf{a}_i\mathbf{F}_i : \mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_m \in \mathbb{R}^n\} \tag{2}$$

spans a subspace of second-order tensors, and is called the *description space* determined by the set of *constitutive parameters* $\{\mathbf{a}_i\}$.

The phenomenological connection between forces (causes) $\mathbf{F}_i$ and fluxes (effects) $\mathbf{J}_j$ is given by the inner product (B.11)

$$\mathbf{J}_j = \mathbf{a}_j \cdot (\mathbf{a}_i\mathbf{F}_i) = (\mathbf{a}_i \cdot \mathbf{a}_j)\mathbf{F}_i = L_{ij}\mathbf{F}_i. \tag{3}$$

The double inner product (B.22) then gives

$$\begin{aligned} \|\mathsf{R}\|^2 = \mathsf{R} : \mathsf{R} &= (\mathbf{a}_i\mathbf{F}_i) : (\mathbf{a}_j\mathbf{F}_j) = \mathbf{a}_j \cdot (\mathbf{a}_i\mathbf{F}_i) \cdot \mathbf{F}_j \\ &= L_{ij}\mathbf{F}_i \cdot \mathbf{F}_j = \mathbf{J}_i \cdot \mathbf{F}_j = \delta \geq 0, \end{aligned} \tag{4}$$

where δ is the dissipation function.

This phenomenlogy describes *isotropic* systems. In “straight” relationships,

$$\mathbf{J}_\alpha = L_{\alpha\alpha}\mathbf{F}_\alpha \tag{5}$$

(Greek subscripts imply no sum: this is the αth term only.) Since $L_{\alpha\alpha}$ is a scalar, (5) says that $\mathbf{J}_\alpha$ is a scalar multiple of $\mathbf{F}_\alpha$, thus the flux is collinear with force. Likewise, in “coupled” relationships,

$$\mathbf{J}_\alpha = L_{\alpha\beta}\mathbf{F}_\beta, \tag{6}$$

the flux is still collinear with force. This is a consequence of the phenomenological connections (3). When we have, in essence,

$$\mathbf{J} = \mathbf{b} \cdot (\mathbf{a}\mathbf{F}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{F} = L\mathbf{F}, \tag{7}$$

the multiplication by the scalar coefficient L is inherent.

For anisotropic systems, the flux is not aligned with the force. The phenomenological connection between them cannot be simple scalar multiplication. This means the vector components of $\mathbf{F}$ must be "independently scaled". We may consider the possibility of a Riemannian description space (which we discussed in Richardson et al. (1982), Richardson and Louie (1986), and Louie and Richardson (1986)), where the constitutive parameters depend upon the forces, viz. $\mathbf{a} = \mathbf{a(F)}$. But as long as the constitutive parameters are vectors, their inner product $L(\mathbf{F}) = \mathbf{a(F)} \cdot \mathbf{b(F)}$ will remain a scalar, albeit in this case a force-dependent one.

To "independently scale" the force components, the cause-to-effect map may be represented by a second-order tensor, so that

$$\mathbf{J} = \mathsf{L} \cdot \mathbf{F}. \tag{8}$$

How may such a tensor L be constructed in our phenomenological calculus? One way to generate a second-order tensor "coefficient" L is that the constitutive parameters themselves are tensors.

4. TRIADIC RESPONSE TENSORS

So let us fix second-order tensors $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_m$, and allow vectors $\mathbf{F}_1$, $\mathbf{F}_2$, …, $\mathbf{F}_m$ to vary. A *(triadic) response tensor* may be defined as the (implied) sum of m triads

$$\underline{\mathsf{R}} = \mathsf{A}_i\mathbf{F}_i. \tag{9}$$

The set

$$D = \{\underline{\mathsf{R}} = \mathsf{A}_i\mathbf{F}_i : \mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_m \in \mathbb{R}^n\} \tag{10}$$

then spans a subspace of *third*-order tensors, and is called the (*triadic) description space* determined by the set of *constitutive parameters* $\{\mathsf{A}_i\}$.

It would then seem reasonable to define the phenomenological connection between forces $\mathbf{F}_i$ and fluxes $\mathbf{J}_j$ by the double inner product (B.29)

$$\mathbf{J}_j = \mathsf{A}_j : (\mathsf{A}_i\mathbf{F}_i) = (\mathsf{A}_i : \mathsf{A}_j)\mathbf{F}_i = L_{ij}\mathbf{F}_i. \tag{11}$$

The triple inner product (B.33) then gives

$$\begin{aligned} \|\underline{\mathsf{R}}\|^2 = \underline{\mathsf{R}} \,\vdots\, \underline{\mathsf{R}} = (\mathsf{A}_i\mathbf{F}_i) \,\vdots\, (\mathsf{A}_j\mathbf{F}_j) = (\mathsf{A}_i : \mathsf{A}_j)(\mathbf{F}_i \cdot \mathbf{F}_j) \\ = L_{ij}\mathbf{F}_i \cdot \mathbf{F}_j = \mathbf{J}_j \cdot \mathbf{F}_j = \delta \geq 0. \end{aligned} \tag{12}$$

But the double inner product of two second-order tensors $L_{ij} = \mathsf{A}_i : \mathsf{A}_j$ is, alas, a scalar. So we still have the collinearity of $\mathbf{J}$ and $\mathbf{F}$.

Next, we may consider the functional definition of the (single) inner product (B.14) of two second-order tensors, and define a new product $\triangleleft$ between a dyad and a triadic this way:

$$\mathbf{J}_j = \mathbf{A}_j \triangleleft (\mathbf{A}_i\mathbf{F}_i) = \mathbf{A}_i \cdot (\mathbf{A}_j \cdot \mathbf{F}_i) = (\mathbf{A}_i \cdot \mathbf{A}_j) \cdot \mathbf{F}_i = \mathbf{L}_{ij} \cdot \mathbf{F}_i. \tag{13}$$

But, unfortunately with this definition, the resulting "dissipation" is not positive definite; i.e. it may happen that

$$(\mathbf{A}_i \cdot \mathbf{A}_j) \cdot \mathbf{F}_i \cdot \mathbf{F}_j = \mathbf{J}_j \cdot \mathbf{F}_j = \delta \ngeq 0. \tag{14}$$

So, it appears that we would not find what we seek from the "standard" inner products from Appendix B. Our answers lie elsewhere, and we shall, therefore, construct our own

$$\mathbf{L}_{ij} = \mathbf{A}_i \diamond \mathbf{A}_j. \tag{15}$$

Once we have a good definition of our new inner product $\diamond$ we may then hierarchically define two more new inner products $\triangleleft$ and $\triangleright\triangleleft$ as

$$\mathbf{J}_j = \mathbf{A}_j \triangleleft (\mathbf{A}_i\mathbf{F}_i) = (\mathbf{A}_i \diamond \mathbf{A}_j) \cdot \mathbf{F}_i = \mathbf{L}_{ij} \cdot \mathbf{F}_i, \tag{16}$$

and

$$\begin{aligned} \|\underline{\mathbf{R}}\|^2 &= \underline{\mathbf{R}} \triangleright\triangleleft \underline{\mathbf{R}} = (\mathbf{A}_i\mathbf{F}_i) \triangleright\triangleleft (\mathbf{A}_j\mathbf{F}_j) = \mathbf{A}_j \triangleleft (\mathbf{A}_i\mathbf{F}_i) \cdot \mathbf{F}_j \\ &= (\mathbf{A}_i \diamond \mathbf{A}_j) \cdot \mathbf{F}_i \cdot \mathbf{F}_j = \mathbf{L}_{ij} \cdot \mathbf{F}_i \cdot \mathbf{F}_j = \mathbf{J}_j \cdot \mathbf{F}_j = \delta. \end{aligned} \tag{17}$$

5. ANISOTROPIC INNER PRODUCTS

In this section when we construct our own inner products, we shall reduce the confusing plethora of subscripts by defining

$$\mathbf{L} = \mathbf{A} \diamond \mathbf{B}, \tag{18}$$

$$\mathbf{J} = \mathbf{B} \triangleleft (\mathbf{A}\mathbf{F}) = (\mathbf{A} \diamond \mathbf{B}) \cdot \mathbf{F} = \mathbf{L} \cdot \mathbf{F}, \tag{19}$$

and

$$(\mathbf{A}\mathbf{F}) \triangleright\triangleleft (\mathbf{B}\mathbf{G}) = \mathbf{B} \triangleleft (\mathbf{A}\mathbf{F}) \cdot \mathbf{G} = (\mathbf{A} \diamond \mathbf{B}) \cdot \mathbf{F} \cdot \mathbf{G} = \mathbf{L} \cdot \mathbf{F} \cdot \mathbf{G}. \tag{20}$$

The ij subscripts of the triadic response tensors in description space may subsequently be reintroduced when required.

Note that the inner product $\diamond$ in (18) is a "single" inner product between two second-order tensors $\mathbf{A}$ and $\mathbf{B}$, and the tensorial action $2 + 2 - 2 = 2$ yields the second-order tensor $\mathbf{L}$. The inner product $\triangleleft$ in (19) a "double" inner product of a second-order tensor $\mathbf{B}$ with a third-order tensor, the triad $\mathbf{AF}$, and the tensorial action $2 + 3 - 2 \times 2 = 1$ yields the vector $\mathbf{J}$. The inner product $\bowtie$ in (20) is a "triple" inner product of two third-order tensors, the triads $\mathbf{AF}$ and $\mathbf{BG}$, and the tensorial action $3 + 3 - 3 \times 2 = 0$ yields a scalar. The sequential and hierarchical nature of the three definitions is such that once $\diamond$ is constructed, the other two, $\triangleleft$ and $\bowtie$, are automatically entailed.

To reduce verbosity, in the remainder of this section we shall use "tensor" to mean "second-order tensor", unless otherwise specified.

Let us now examine what the requirements are for our new inner product $\diamond$. First, we need the dissipation function to be positive definite. So we have to have

$$(\mathbf{AF}) \bowtie (\mathbf{AF}) = (\mathbf{A} \diamond \mathbf{A}) \cdot \mathbf{F} \cdot \mathbf{F} = \mathbf{L} \cdot \mathbf{F} \cdot \mathbf{F} = \mathbf{J} \cdot \mathbf{F} = \delta \geq 0 \tag{21}$$

(with equality iff $\mathbf{AF} = \mathbf{0}$). It means $\mathbf{A} \diamond \mathbf{A}$ itself has to be positive definite. This, in particular, implies that the signature of $\mathbf{A} \diamond \mathbf{A}$ be n, the dimension of the base space $\mathbb{R}^n$, whence the rank of $\mathbf{A} \diamond \mathbf{A}$ must also be n (i.e. non-degeneracy, full rank). (See Louie and Richardson (1986) for a discussion of these bilinear form concepts.)

The second requirement that we need our new inner product $\diamond$ to satisfy is the continuity of transition between isotropic and anisotropic systems. In other words, the transition between $\mathbf{J} = L\mathbf{F}$ and $\mathbf{J} = \mathbf{L} \cdot \mathbf{F}$ must happen is such away that a "jump discontinuity", a situation analogous to the Gibbs paradox, does *not* occur. (Incidentally, our phenomenological calculus completely resolves the classical Gibbs paradox. See Richardson (1989).) In an isotropic system, the constitutive parameter is a vector $\mathbf{a}$. But it may be represented as the diagonal tensor $\mathbf{D}[\mathbf{a}]$, the diagonal entries of which are the components of the vector $\mathbf{a}$; viz.

$$\mathbf{D}[\mathbf{a}]_{ij} = \begin{cases} a_i, & i = j \\ 0, & i \neq j \end{cases} . \tag{22}$$

The diagonal tensor is a most natural representation of isotropy. Indeed, an anisotropic system becomes isotropic precisely when its tensorial constitutive parameters $\mathbf{A}$ have zeroes in all off-diagonal entries. Also, the inner product $L = \mathbf{a} \cdot \mathbf{b}$ may be represented as the diagonal tensor $L\mathbf{I} = (\mathbf{a} \cdot \mathbf{b})\mathbf{I}$ (where $\mathbf{I}$ is the identity tensor); viz.

$$[L\mathbf{I}]_{ij} = [(\mathbf{a} \cdot \mathbf{b})\mathbf{I}]_{ij} = \begin{cases} L, & i = j \\ 0, & i \neq j \end{cases} . \tag{23}$$

Then the cause-to-effect map in an isotropic system may be reset in the same tensor-vector inner product form as in an anisotropic system:

$$\mathbf{J} = (L\mathbf{I}) \cdot \mathbf{F}. \tag{24}$$

Thus, if we can construct $\mathbf{A} \diamond \mathbf{B}$ in such a way that when $\mathbf{A} = \mathbf{D}[\mathbf{a}]$ and $\mathbf{B} = \mathbf{D}[\mathbf{b}]$,

$$\mathbf{L} = \mathbf{D}[\mathbf{a}] \diamond \mathbf{D}[\mathbf{b}] = L\mathbf{I} = (\mathbf{a} \cdot \mathbf{b})\mathbf{I}, \tag{25}$$

then we have assured the continuity of the isotropy–anisotropy transition.

The construction of a bilinear form $\mathbf{A} \diamond \mathbf{B}$ that satisfies (21) and (25) is quite a complicated process. First, define the *row rotation tensor* $\mathbf{R}[k]$, for $k = 1, 2, \ldots, n$, as

$$[\mathbf{R}[k]]_{ij} = \begin{cases} 1, & \text{if } i - j + 1 = k \pmod{n} \\ 0, & \text{otherwise} \end{cases} \tag{26}$$

(All the arithmetic in the subscripts are in modulo n, with the identification $0 = n$.) For example, when $n = 6$ and $k = 3$ (We shall use these values in a running example.), we have

$$\mathbf{R}[3] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{27}$$

One easily verifies that

$$\mathbf{R}[1] = \mathbf{I}, \tag{28}$$

and

$$\mathbf{R}[k] \cdot \mathbf{R}[l] = \mathbf{R}[k + l - 1], \tag{29}$$

whence

$$\mathbf{R}[k] \cdot \mathbf{R}[n - k + 2] = \mathbf{R}[n + 1] = \mathbf{R}[1] = \mathbf{I}. \tag{30}$$

One may likewise verify that

$$\mathbf{R}[k]^T = \mathbf{R}[n-k+2], \tag{31}$$

and therefore

$$\mathbf{R}[k]^{-1} = \mathbf{R}[k]^T; \tag{32}$$

i.e., $\mathbf{R}[k]$ *is orthogonal.*

The action of $\mathbf{R}[k]$ is such that "left multiplication" to a tensor $\mathbf{A}$ rotates the rows of $\mathbf{A}$ cyclically $k-1$ steps. In other words, the rows are mapped thus:

$$\begin{aligned}&(1,2,\ldots,n-k+l,\ n-k+2,\ldots,n)\\ &\quad\mapsto(k,k+1,\ldots,n,\ 1,\ldots,k-1);\end{aligned} \tag{33}$$

i.e., row i is mapped to row $i+k-1$ (mod n). For example,

$$\mathbf{R}[5]\cdot\mathbf{A} = \begin{bmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\end{bmatrix}\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}&a_{15}&a_{16}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{25}&a_{26}\\a_{31}&a_{32}&a_{33}&a_{34}&a_{35}&a_{36}\\a_{41}&a_{42}&a_{43}&a_{44}&a_{45}&a_{46}\\a_{51}&a_{52}&a_{53}&a_{54}&a_{55}&a_{56}\\a_{61}&a_{62}&a_{63}&a_{64}&a_{65}&a_{66}\end{bmatrix}$$

$$= \begin{bmatrix}a_{31}&a_{32}&a_{33}&a_{34}&a_{35}&a_{36}\\a_{41}&a_{42}&a_{43}&a_{44}&a_{45}&a_{46}\\a_{51}&a_{52}&a_{53}&a_{54}&a_{55}&a_{56}\\a_{61}&a_{62}&a_{63}&a_{64}&a_{65}&a_{66}\\a_{11}&a_{12}&a_{13}&a_{14}&a_{15}&a_{16}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{25}&a_{26}\end{bmatrix}. \tag{34}$$

"Right multiplication" by $\mathbf{R}[k]^{-1}(=\mathbf{R}[k]^T)$ to a tensor $\mathbf{A}$ rotates the columns of $\mathbf{A}$ cyclically $k-1$ steps; i.e. the columns are mapped as in (33), with column j mapped to column $j+k-1$ (mod n). So the combined action, the "similarity transform"

$$\mathbf{A}[k] = \mathbf{R}[k]\cdot\mathbf{A}\cdot\mathbf{R}[k]^{-1}, \tag{35}$$

moves the entry a_{11} to position (k,k) (equivalently, the entry $a_{(n-k+2)(n-k+2)}$ to position (1,1)), with the other entries rotated cyclically. A little bit of algebra reveals the components as

$$[\mathbf{A}[k]]_{ij} = A_{(i-k+1)(j-k+1)} \tag{36}$$

(with no sum over k here, of course, and the arithmetic in the subscripts is in modulo n). For example,

$$\mathbf{A}[5] = \mathbf{R}[5] \cdot \mathbf{A} \cdot \mathbf{R}[5]^{-1} = \mathbf{R}[5] \cdot \mathbf{A} \cdot \mathbf{R}[3]$$

$$= \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{33} & a_{34} & a_{35} & a_{36} & a_{31} & a_{32} \\ a_{43} & a_{44} & a_{45} & a_{46} & a_{41} & a_{42} \\ a_{53} & a_{54} & a_{55} & a_{56} & a_{51} & a_{52} \\ a_{63} & a_{64} & a_{65} & a_{66} & a_{61} & a_{62} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{11} & a_{12} \\ a_{23} & a_{24} & a_{25} & a_{26} & a_{21} & a_{22} \end{bmatrix}. \tag{37}$$

It is clear that $\mathbf{A}[1] = \mathbf{R}[1] \cdot \mathbf{A} \cdot \mathbf{R}[1]^{-1} = \mathbf{I} \cdot \mathbf{A} \cdot \mathbf{I} = \mathbf{A}$.

We are now ready to define the inner product $\mathbf{A} \diamond \mathbf{B}$ as the *sum* of n inner products of two tensors:

$$\begin{aligned} \mathbf{A} \diamond \mathbf{B} &= \mathbf{A}[k] \cdot \mathbf{B}[k]^T = \mathbf{A}[k] \cdot \mathbf{B}^T[k] \\ &= (\mathbf{R}[k] \cdot \mathbf{A} \cdot \mathbf{R}[k]^{-1}) \cdot (\mathbf{R}[k] \cdot \mathbf{B}^T \cdot \mathbf{R}[k]^{-1}) \\ &= \mathbf{R}[k] \cdot (\mathbf{A} \cdot \mathbf{B}^T) \cdot \mathbf{R}[k]^{-1} \end{aligned} \tag{38}$$

(sum over k from 1 to n). Note that $\mathbf{A} \cdot \mathbf{B}^T$ appears in the double inner product (B.18) of two tensors (with the trace function completing the map to scalars). The form $\mathbf{A} \cdot \mathbf{A}^T$ is positive definite, this being a theorem in linear algebra. It also follows from theorems in linear algebra that positive definiteness is invariant under similarity, and that sums of positive definite forms are positive definite. Thus our inner product defined in (38) satisfies the positive definiteness requirement (21).

In order to verify that our inner product $\mathbf{A} \diamond \mathbf{B}$ also satisfies the continuity requirement (25), we need to discover what the components of $\mathbf{A} \diamond \mathbf{B}$ look like, First, note that

$$[\mathbf{A}\cdot\mathbf{B}^T]_{ij} = A_{il}B_{jl} \text{ (sum over } l). \tag{39}$$

Then (39) gives, for each k,

$$[\mathbf{R}[k]\cdot(\mathbf{A}\cdot\mathbf{B}^T)\cdot\mathbf{R}[k]^{-1}]_{ij} = A_{(i-k+1)l}\,B_{(j-k+1)l} \quad \text{(sum over } l, \text{ but no sum over } k \text{ yet).} \tag{40}$$

Finally, when we sum (40) over k, we have

$$[\mathbf{A}\diamond\mathbf{B}]_{ij} = A_{(i-k+1)l}\,B_{(j-k+1)l} \quad \text{(sum over both } l \text{ and } k). \tag{41}$$

Note that, because of the summing over both l and k, for each ij, the component (41) has n^2 terms in the sum, and each of these terms is the product of one component each from **A** and **B**.

When $i = j$, (41) becomes

$$[\mathbf{A}\diamond\mathbf{B}]_{ii} = A_{(i-k+1)l}\,B_{(i-k+1)l} = A_{kl}\,B_{kl} \quad \text{(sum over both } l \text{ and } k, \text{ but no sum over } i), \tag{42}$$

which is precisely the double inner product $\mathbf{A}:\mathbf{B}$ (cf. (B.20)). In other words, all the diagonal entries of $\mathbf{A}\diamond\mathbf{B}$ are $\mathbf{A}:\mathbf{B}$.

Because of the modulo n nature of the subscript arithmetic, there are actually more duplications of values. Among the n^2 components in (41), there are much fewer distinct scalar values. Let us define, for $i = 1, 2,\ldots, n$, the scalars

$$\mathbf{A}:\mathbf{B}\{i\} = A_{kl}\,B_{(i+k-1)l} \quad \text{(sum over both } l \text{ and } k). \tag{43}$$

The previous paragraphs says $[\mathbf{A}\diamond\mathbf{B}]_{ii} = \mathbf{A}:\mathbf{B} = \mathbf{A}:\mathbf{B}\{1\}$. In fact, all the components of $\mathbf{A}\diamond\mathbf{B}$ are of the form (43):

$$[\mathbf{A}\diamond\mathbf{B}]_{ij} = \mathbf{A}:\mathbf{B}\{j-i+1\} \tag{44}$$

(with $j - i + 1 \pmod n$)). Thus, among the n^2 ij-components in (37), there are only n distinct values $\mathbf{A}:\mathbf{B}\{i\}$. For example, when $n = 6$, there are only 6 distinct values in the 36 components of the tensor:

$$\mathbf{A}\diamond\mathbf{B}=\begin{bmatrix}\mathbf{A}{:}\mathbf{B}\{1\} & \mathbf{A}{:}\mathbf{B}\{2\} & \mathbf{A}{:}\mathbf{B}\{3\} & \mathbf{A}{:}\mathbf{B}\{4\} & \mathbf{A}{:}\mathbf{B}\{5\} & \mathbf{A}{:}\mathbf{B}\{6\}\\ \mathbf{A}{:}\mathbf{B}\{6\} & \mathbf{A}{:}\mathbf{B}\{1\} & \mathbf{A}{:}\mathbf{B}\{2\} & \mathbf{A}{:}\mathbf{B}\{3\} & \mathbf{A}{:}\mathbf{B}\{4\} & \mathbf{A}{:}\mathbf{B}\{5\}\\ \mathbf{A}{:}\mathbf{B}\{5\} & \mathbf{A}{:}\mathbf{B}\{6\} & \mathbf{A}{:}\mathbf{B}\{1\} & \mathbf{A}{:}\mathbf{B}\{2\} & \mathbf{A}{:}\mathbf{B}\{3\} & \mathbf{A}{:}\mathbf{B}\{4\}\\ \mathbf{A}{:}\mathbf{B}\{4\} & \mathbf{A}{:}\mathbf{B}\{5\} & \mathbf{A}{:}\mathbf{B}\{6\} & \mathbf{A}{:}\mathbf{B}\{1\} & \mathbf{A}{:}\mathbf{B}\{2\} & \mathbf{A}{:}\mathbf{B}\{3\}\\ \mathbf{A}{:}\mathbf{B}\{3\} & \mathbf{A}{:}\mathbf{B}\{4\} & \mathbf{A}{:}\mathbf{B}\{5\} & \mathbf{A}{:}\mathbf{B}\{6\} & \mathbf{A}{:}\mathbf{B}\{1\} & \mathbf{A}{:}\mathbf{B}\{2\}\\ \mathbf{A}{:}\mathbf{B}\{2\} & \mathbf{A}{:}\mathbf{B}\{3\} & \mathbf{A}{:}\mathbf{B}\{4\} & \mathbf{A}{:}\mathbf{B}\{5\} & \mathbf{A}{:}\mathbf{B}\{6\} & \mathbf{A}{:}\mathbf{B}\{1\}\end{bmatrix}. \tag{45}$$

When the system is isotropic (i.e. when $A_{ij}=B_{ij}=0$ for $i \neq j$), we readily see from (43) that $\mathbf{A}:\mathbf{B}\{i\}=0$ for $i=2, 3,\ldots, n$, and that

$$\mathbf{A}:\mathbf{B}\{1\}=A_{kk}\,B_{kk}. \tag{46}$$

Thus, when $\mathbf{a}=(A_{11},A_{22},\ldots,A_{nn})$ and $\mathbf{b}=(B_{11},B_{22},\ldots,B_{nn})$, whence $\mathbf{A}=\mathbf{D}[\mathbf{a}]$ and $\mathbf{B}=\mathbf{D}[\mathbf{b}]$, we have

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{A}:\mathbf{B}\{1\}, \tag{47}$$

and hence do indeed have

$$\mathbf{A}\diamond\mathbf{B}=\mathbf{L}=\mathbf{D}[\mathbf{a}]\diamond\mathbf{D}[\mathbf{b}]=(L\mathbf{I})=(\mathbf{a}\cdot\mathbf{b})\mathbf{I}, \tag{48}$$

as required by the isotropy–anisotropy continuity condition (25).

With the components of our $\mathbf{A}\diamond\mathbf{B}$ (18) defined as in (44), the hierarchical definitions (19) and (20) of $\triangleleft$ and $\triangleright\triangleleft$ follow:

$$J_i=[\mathbf{B}\triangleleft(\mathbf{AF})]_i=[(\mathbf{A}\diamond\mathbf{B})\cdot\mathbf{F}]_i=[\mathbf{A}\diamond\mathbf{B}]_{ij}F_j, \tag{49}$$

$$\begin{aligned}(\mathbf{AF})\triangleright\triangleleft(\mathbf{BG}) &= \mathbf{B}\triangleleft(\mathbf{AF})\cdot\mathbf{G}=(\mathbf{A}\diamond\mathbf{B})\cdot\mathbf{F}\cdot\mathbf{G}\\ &= [\mathbf{A}\diamond\mathbf{B}]_{ij}F_jG_i\\ &= [\mathbf{A}\diamond\mathbf{B}]_{ii}F_iG_i+\sum_{j\neq i}[\mathbf{A}\diamond\mathbf{B}]_{ij}F_jG_i\\ &= (\mathbf{A}:\mathbf{B})(\mathbf{F}\cdot\mathbf{G})+\sum_{j\neq i}[\mathbf{A}\diamond\mathbf{B}]_{ij}F_jG_i\\ &= \mathbf{AF}\,\dot{:}\,\mathbf{BG}+\text{other terms}\end{aligned} \tag{50}$$

So we see that our "triple inner product" $(\mathbf{AF})\triangleright\triangleleft(\mathbf{BG})$ is the standard one $\mathbf{AF}\,\dot{:}\,\mathbf{BG}$ "plus a little more".

6. TENSOR SYMMETRY AND ONSAGER RECIPROCITY

Recall that in the definition of our inner product $\mathbf{L} = \mathbf{A} \diamond \mathbf{B}$ of two constitutive parameters $\mathbf{A}$ and $\mathbf{B}$, all three objects $\mathbf{A}, \mathbf{B}$, and $\mathbf{L}$ are second-order tensors. There are two *independent* issues concerning "symmetry" to consider:

(i) the symmetry of $\mathbf{L}$ itself as a second-order tensor: i.e., whether

$$[\mathbf{A} \diamond \mathbf{B}]_{ji} = [\mathbf{A} \diamond \mathbf{B}]_{ij}? \tag{51}$$

(ii) the symmetry of the inner product as a bilinear function: i.e., whether

$$\mathbf{B} \diamond \mathbf{A} = \mathbf{A} \diamond \mathbf{B}? \tag{52}$$

It is evident from definition (41), and also relation (44), that $\mathbf{L} = \mathbf{A} \diamond \mathbf{B}$ itself as a tensor is *not* symmetric; i.e., in general.

$$[\mathbf{A} \diamond \mathbf{B}]_{ji} \neq [\mathbf{A} \diamond \mathbf{B}]_{ij}. \tag{53}$$

The tensor $\mathbf{A} \diamond \mathbf{A}$, however, is symmetric:

$$\begin{aligned}
[\mathbf{A} \diamond \mathbf{A}]_{ji} &= \mathbf{A} : \mathbf{A}\{i - j + 1\} && [(44)] \\
&= A_{kl} A_{((i-j+1)+k-1)l} && [(43)] \\
&= A_{kl} A_{(i-j+k)l} \\
&= A_{((j-i)+k)l} A_{((j-i)+i-j+k)l} && [\dagger] \\
&= A_{(j-i+k)l} A_{kl} \\
&= A_{kl} A_{(j-i+k)l} && [\ddagger] \\
&= A_{kl} A_{((j-i+1)+k-1)l} \\
&= \mathbf{A} : \mathbf{A}\{j - i + 1\} && [(43)] \\
&= [\mathbf{A} \diamond \mathbf{A}]_{ij} && [(44)]
\end{aligned} \tag{54}$$

The transition † holds because as k ranges from 1 to n, so does $h+k$ for any constant h. A requirement in the sum $A_{kl}A_{(i-j+k)l}$ is that the first index of the second subscript be $i-j$ more than the first index of the first subscript. So one may replace $\{k, i-j+k\}$ by $\{(j-i)+k, (j-i)+i-j+k\}$ and the sum remains the same. The commutativity transition ‡ is only applicable in $\mathbf{A} \diamond \mathbf{A}$, but not in $\mathbf{A} \diamond \mathbf{B}$.

The symmetry $[\mathbf{A} \diamond \mathbf{A}]_{ji} = [\mathbf{A} \diamond \mathbf{A}]_{ij}$ further reduces the number of distinct components in the tensor $\mathbf{A} \diamond \mathbf{A}$. We have

$$\begin{aligned} [\mathbf{A} \diamond \mathbf{A}]_{ji} &= \mathbf{A} : \mathbf{A}\{i - j + 1\} && [(44)] \\ &= \mathbf{A} : \mathbf{A}\{n + i - j + 1\} && [\text{mod } n] \\ &= \mathbf{A} : \mathbf{A}\{n - (j - i + 1) + 2\} \\ &= [\mathbf{A} \diamond \mathbf{A}]_{ij} = \mathbf{A} : \mathbf{A}\{j - i + 1\}. && [(54)] \end{aligned} \tag{55}$$

So from $\mathbf{A} : \mathbf{A}\{n - (j - i + 1) + 2\} = \mathbf{A} : \mathbf{A}\{j - i + 1\}$ for all ij in (55), we obtain an additional cyclic symmetry:

$$\mathbf{A} : \mathbf{A}\{i\} = \mathbf{A} : \mathbf{A}\{n - i + 2\}. \tag{56}$$

The components of $\mathbf{A} \diamond \mathbf{A}$ may be rewritten as

$$[\mathbf{A} \diamond \mathbf{A}]_{ij} = \begin{cases} \mathbf{A} : \mathbf{A}\{|j - i| + 1\}, & \text{if } |j - i| \le [n/2] \\ \mathbf{A} : \mathbf{A}\{n - |j - i| + 1\}, & \text{otherwise} \end{cases}, \tag{57}$$

(where [ ] is the integer-part function, whence $[2m/2] = [(2m+1)/2] = m$). Thus, among the n^2 components in (37), there are only $[n/2] + 1$ distinct values $\mathbf{A} : \mathbf{A}\{i\}$. For example, when $n = 6$, there are only four distinct values in the 36 components of the symmetric tensor:

$$\mathbf{A} \diamond \mathbf{A} = \begin{bmatrix} \mathbf{A}:\mathbf{A}\{1\} & \mathbf{A}:\mathbf{A}\{2\} & \mathbf{A}:\mathbf{A}\{3\} & \mathbf{A}:\mathbf{A}\{4\} & \mathbf{A}:\mathbf{A}\{3\} & \mathbf{A}:\mathbf{A}\{2\} \\ \mathbf{A}:\mathbf{A}\{2\} & \mathbf{A}:\mathbf{A}\{1\} & \mathbf{A}:\mathbf{A}\{2\} & \mathbf{A}:\mathbf{A}\{3\} & \mathbf{A}:\mathbf{A}\{4\} & \mathbf{A}:\mathbf{A}\{3\} \\ \mathbf{A}:\mathbf{A}\{3\} & \mathbf{A}:\mathbf{A}\{2\} & \mathbf{A}:\mathbf{A}\{1\} & \mathbf{A}:\mathbf{A}\{2\} & \mathbf{A}:\mathbf{A}\{3\} & \mathbf{A}:\mathbf{A}\{4\} \\ \mathbf{A}:\mathbf{A}\{4\} & \mathbf{A}:\mathbf{A}\{3\} & \mathbf{A}:\mathbf{A}\{2\} & \mathbf{A}:\mathbf{A}\{1\} & \mathbf{A}:\mathbf{A}\{2\} & \mathbf{A}:\mathbf{A}\{3\} \\ \mathbf{A}:\mathbf{A}\{3\} & \mathbf{A}:\mathbf{A}\{4\} & \mathbf{A}:\mathbf{A}\{3\} & \mathbf{A}:\mathbf{A}\{2\} & \mathbf{A}:\mathbf{A}\{1\} & \mathbf{A}:\mathbf{A}\{2\} \\ \mathbf{A}:\mathbf{A}\{2\} & \mathbf{A}:\mathbf{A}\{3\} & \mathbf{A}:\mathbf{A}\{4\} & \mathbf{A}:\mathbf{A}\{3\} & \mathbf{A}:\mathbf{A}\{2\} & \mathbf{A}:\mathbf{A}\{1\} \end{bmatrix}. \tag{58}$$

On the issue of the symmetry of the inner product, we have (sum over k)

$$\begin{aligned}
\mathbf{B} \diamond \mathbf{A} &= \mathbf{R}[k] \cdot (\mathbf{B} \cdot \mathbf{A}^T) \cdot \mathbf{R}[k]^{-1} && [(38)] \\
&= \left\{ \left[\mathbf{R}[k] \cdot (\mathbf{B} \cdot \mathbf{A}^T) \cdot \mathbf{R}[k]^{-1} \right]^T \right\}^T \\
&= \left\{ \left[\mathbf{R}[k]^{-1} \right]^T \cdot \mathbf{A} \cdot \mathbf{B}^T \cdot \mathbf{R}[k]^T \right\}^T \\
&= \left\{ \left[\mathbf{R}[k]^T \right]^{-1} \cdot (\mathbf{A} \cdot \mathbf{B}^T) \cdot \mathbf{R}[k]^T \right\}^T \\
&= \left\{ \mathbf{R}[k] \cdot (\mathbf{A} \cdot \mathbf{B}^T) \cdot \mathbf{R}[k]^{-1} \right\}^T && [\S] \\
&= (\mathbf{A} \diamond \mathbf{B})^T
\end{aligned} \tag{59}$$

The transition § holds because as k ranges from 1 to n, the pair of tensors $\left\{ \left[\mathbf{R}[k]^T \right]^{-1}, \mathbf{R}[k]^T \right\}$ "bracketing" $\mathbf{A} \cdot \mathbf{B}^T$ ranges through the same set as the pair $\{\mathbf{R}[k], \mathbf{R}[k]^{-1}\}$. Thus the latter may replace the former and the sum remains the same. So the inner product $\diamond$ is not symmetric either, but is, rather "transpose symmetric". While $\diamond$ is an "inner product" in the sense that it is a contraction, it also maintains this "transpose symmetric" character of an outer product (cf.(B.5)). Note that (53) and (59) say, respectively, $\mathbf{A} \diamond \mathbf{B} \neq (\mathbf{A} \diamond \mathbf{B})^T$ and $\mathbf{B} \diamond \mathbf{A} = (\mathbf{A} \diamond \mathbf{B})^T$.

In the indicial notation of triadic description space, let $\mathbf{L}_{ij} = \mathbf{A}_i \diamond \mathbf{A}_j$ and $\mathbf{L}_{ji} = \mathbf{A}_j \diamond \mathbf{A}_i$. Then (59) says the two tensors are transpose of each other

$$\mathbf{L}_{ji} = (\mathbf{L}_{ij})^T. \tag{60}$$

This is the Onsager reciprocity in an anisotropic system. Note that the corresponding relationship in our phenomenological calculus of isotropic systems is

$$L_{ji} = L_{ij}. \tag{61}$$

Our relation (60) is *not* a contradiction to (61). Relation (61) says that those pairs of scalars are equal. When an anisotropic system is taken to the limit of an isotropic system, we have (25)

$$\mathbf{L}_{ij} = (\mathbf{a}_i \cdot \mathbf{a}_j)\mathbf{I}. \tag{62}$$

Trivially, the transpose of a *diagonal* tensor is itself, whence

$$\mathbf{L}_{ji} = (\mathbf{a}_j \cdot \mathbf{a}_i)\mathbf{I} = (\mathbf{a}_i \cdot \mathbf{a}_j)\mathbf{I} = \mathbf{L}_{ij} = (\mathbf{L}_{ij})^T. \tag{63}$$

Alternatively, we may say that (61) is simply a special instance of (60), since the "transpose" of a scalar is the same scalar itself.

It is clear that $\mathbf{B} \triangleleft (\mathbf{AF})$ itself as a second-order tensor is not symmetric, and that there can be no symmetry in the inner product $\triangleleft$, since it is between tensors of different orders.

The scalar inner product $\triangleright\triangleleft$ is, however, symmetric:

$$\begin{aligned}(\mathbf{AF}) \triangleright\triangleleft (\mathbf{BG}) &= [\mathbf{A} \diamond \mathbf{B}]_{ij} F_j G_i \qquad [(50)] \\ &= [\mathbf{B} \diamond \mathbf{A}]_{ji} F_j G_i \qquad [(59)] \\ &= [\mathbf{B} \diamond \mathbf{A}]_{ij} G_j F_i \qquad [\,¤\,] \\ &= (\mathbf{BG}) \triangleright\triangleleft (\mathbf{AF}). \quad [(50)]\end{aligned} \tag{64}$$

The transition ¤ holds because here i and j are just "dummy" indices of summation from 1 to n, and so may be replaced by any symbols. Further, it follows from the positive definiteness requirement (21) that $\bowtie$ is also positive definite; i.e. $\bowtie$ is an inner product in the sense of a positive definite symmetric bilinear form. We have, therefore, established that our triadic description space with $\bowtie$ is an inner product space.

7. DUALITY

Let us now consider the concept of dual representation for response tensors (cf. Appendix A, (A.8)ff.).

For *dyadic* response tensors, we saw that it would usually take additional information to go from effect to cause. This was discussed in detail in Richardson et al. (1982), and also in Louie et al. (1982). The effect-to-cause reversal hinges on the invertibility of the $m \times m$ Gram matrix (L^{ij}) of $\{\mathbf{a}^i\}$. The general non-invertibility of the Gram matrix is the reason why the L^{ij} arrows in the DID (A.11) are unidirectional. Recall that the rank k of the Gram matrix is the same as the dimension of the subspace spanned by $\{\mathbf{a}^i\}$ hence $k \leq m$ and $k \leq n$, and that (L^{ij}) is invertible if and only if $k = m$.

The "linear system of equations" (A.9) (with the vector notation $\mathbf{a}$ instead of the general Hilbert space a)

$$\mathbf{a}_j L^{ij} = \mathbf{a}^i \tag{65}$$

is in fact the matrix equation

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_m]\begin{bmatrix} L^{11} & L^{21} & \dots & L^{m1} \\ L^{12} & L^{22} & \dots & L^{m2} \\ \vdots & \vdots & \ddots & \vdots \\ L^{1m} & L^{2m} & \dots & L^{mm} \end{bmatrix} = [\mathbf{a}^1 \ \mathbf{a}^2 \ \dots \ \mathbf{a}^m] \tag{66}$$

where the first matrix is the $n \times m$ matrix with the components of the m dual constitutive parameters $\mathbf{a}_j \in \mathbb{R}^n$ as columns, the second is the $m \times m$ transpose of the Gram matrix, $(L^{ji}) = (L^{ij})^T$ (but because of the symmetry $L^{ji} = L^{ij}$ the distinction is not crucial here), and the third is the $n \times m$ matrix with the m constitutive parameters $\mathbf{a}^i \in \mathbb{R}^n$ as columns. The dual representation (A.10)

$$L^{ij}\mathbf{a}_j = \mathbf{a}^i \tag{67}$$

is the matrix equation

$$\begin{bmatrix} L^{11} & L^{12} & \dots & L^{1m} \\ L^{21} & L^{22} & \dots & L^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ L^{m1} & L^{m2} & \dots & L^{mm} \end{bmatrix}\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix}, \tag{68}$$

where the three matrices are $m \times m$, $m \times n$, and $m \times n$ respectively, with the components of the m constitutive parameters as rows in the latter two matrices.

For *triadic* response tensors, the situation is a little more complex. In the dual representation

$$\underline{\mathbf{R}} = \mathbf{A}_i\mathbf{F}_i = \mathbf{B}_j\mathbf{J}_j \tag{69}$$

the effect-to-cause phenomenology would require an invertible map from a collection of second-order tensors to another. The linear system of equations corresponding to (65) is

$$\mathbf{B}_j\mathbf{L}_{ij} = \mathbf{A}_i, \tag{70}$$

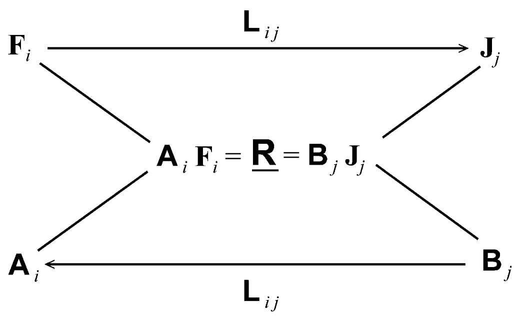
and corresponding to (66) is the *block matrix equation*

$$[\mathbf{B}_1 \ \mathbf{B}_2 \ \dots \ \mathbf{B}_m]\begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{21} & \dots & \mathbf{L}_{m1} \\ \mathbf{L}_{12} & \mathbf{L}_{22} & \dots & \mathbf{L}_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{L}_{1m} & \mathbf{L}_{2m} & \dots & \mathbf{L}_{mm} \end{bmatrix} = [\mathbf{A}_1 \ \mathbf{A}_2 \ \dots \ \mathbf{A}_m]. \tag{71}$$

Each of $\mathbf{A}$ and $\mathbf{B}$, being a second-order tensor, is an $n \times n$ matrix. The first and third block matrices in (71) are, therefore, of dimension $n \times mn$. Each $\mathbf{L}$ is also a second-order tensor, hence an $n \times n$ matrix. The middle block matrix in (71) is, therefore, of dimension $mn \times mn$. It is the invertibility of this "transpose Gram block matrix" $(\mathbf{L}_{ij})$ that determines the effect-to-cause reversal of a triadic description space. (The triadic equations that would correspond to the dual forms (67) and (68) would involve transposes of the blocks themselves, because of the transpose symmetry $\mathbf{L}_{ji} = (\mathbf{L}_{ij})^T$ instead of the symmetry $L^{ji} = L^{ij}$.)

Analogous to the dyadic case, here the rank k of the Gram matrix is the same as the dimension of the subspace (of second-order tensors) spanned by $\{\mathbf{A}_i\}$, hence $k \leq mn$ and $k \leq n^2$; $(\mathbf{L}_{ij})$ is invertible if and only if $k = mn$.

The corresponding duality-invariance diagram (DID) would look like this:



(72)

APPENDIX A. DESCRIPTION SPACE AND THE PHENOMENOLOGICAL CALCULUS POSTULATES

Let H be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$. Let H^* be its dual space, i.e. the space of all continuous linear functionals on H, with corresponding dual inner product $\langle \cdot , \cdot \rangle^*$. A type-(1,1) tensor on H is a map from $H^* \times H$ to $\mathbb{R}$ which is bilinear, i.e. linear in each of its two arguments. The linear space of type-(1,1) tensors on H is denoted $T^1_1(H)$.

A *dyad* aF, where $a \in H^*$ and $F \in H$, is a special type-(1,1) tensor, with its actions defined by

$$aF(b, G) = \langle a, b \rangle^* \langle F, G \rangle \tag{A.1}$$

for $b \in H^*$, $G \in H$. Given dyads aF and bG, their *double inner product* is defined via (A.1), viz.

$$\langle\langle aF, bG\rangle\rangle = \langle a, b\rangle^* \langle F, G\rangle. \tag{A.2}$$

A *dyadic* is a finite sum of dyads, whence

$$\mathbf{R} = a^i F_i, \tag{A.3}$$

where, $a^i \in H^*$ and $F_i \in H$. A modified Einstein summation convention is used: repeated *upper and lower* Roman indices denote summation, thus

$$a^i F_i = a^1 F_1 + a^2 F_2 + \cdots + a^m F_m; \tag{A.4}$$

repeated Greek indices denote a particular term, thus $a^\alpha F_\alpha$ is the αth term only. Clearly $\mathbf{R} \in T^1_1(H)$, and the definition of the double inner product may be extended to dyadics. In Louie et al. (1982), we showed that every type-(1,1) tensor might be represented as a dyadic, and so the double inner product is in fact well defined on all of $T^1_1(H)$.

For fixed covariant vectors a^1, a^2,..., $a^m \in H^*$, the dyadics $\mathbf{R} = a^i F_i$ (with F_1, F_2, ..., F_m ranging freely in H) span a linear subspace of $T^1_1(H)$. We call this set

$$D = \{\mathbf{R} = a^i F_i : F_i \in H\} \tag{A.5}$$

the *description space* determined by the set of *constitutive parameters* $\{a^i\}$. Members $\mathbf{R}$ of D are called *response tensors*. Thus response tensors are simply dyadics having a specified form $\mathbf{R} = a^i F_i$ for fixed constitutive parameters $\{a^i\}$, and belonging to a specified subspace D of $T^1_1(H)$. While $T^1_1(H)$ itself may not be a Hilbert space, the description space is complete with respect to the norm associated with the double inner product; thus D is a Hilbert space.

The metric structure of our phenomenological calculus is formally defined by these three postulates (which first appeared in Richardson et al. (1982)):

Postulate G1. A complex system is characterized by a set of vectors $\{a^i\}$ that depends on the constitution of the system. A system response is then characterized by the specification of the causal action F_i, and of those constitutive properties a^i of the system which are agents of mediation between action and response. The index i denotes (functional) subsystems.

Postulate G2. The system dynamics are characterized phenomenologically by the response tensor.

Postulate G3. The response tensor is invariant under coordinate transformations in the description space.

The phenomenological connection between causes and effects is provided by the geometric structure of description spaces. We *define* effects J^j to be the duals of the causes F_i, using the fact that $\mathbf{R} = a^i F_i$ is, among other things, a linear mapping $H^* \to H^*$, defined by

$$\mathbf{R}(a^j, \cdot) = (a^i F_i)(a^j, \cdot) = \langle a^i, a^j \rangle^* F_i = L^{ij} F_i \equiv J^j \tag{A.6}$$

with phenomenological coefficients

$$L^{ij} = \langle a^i, a^j \rangle^*. \tag{A.7}$$

(Readers familiar with irreversible thermodynamics may have recognized that it and our bilinear phenomenology have a similar metric structure. Indeed the latter was motivated by the former. But our phenomenology is a great deal more general)

Writing $\mathbf{R}$ in its dual representation

$$\mathbf{R} = a^i F_i = a_j J^j \tag{A.8}$$

leads to the linear system of equations

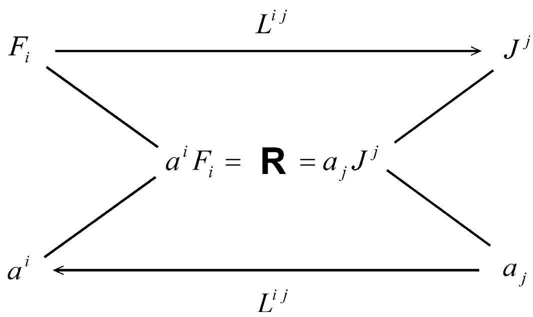
$$a_j L^{ij} = a^i, \tag{A.9}$$

or equivalently

$$L^{ij} a_j = a^i, \tag{A.10}$$

for the dual constitutive parameters $a_1, a_2, \ldots, a_m \in H$. The set $\{a_j\}$ would be determined uniquely if the Gram matrix (L^{ij}) of $\{a^i\}$ were invertible. But this in general is not the case, and we have some degrees of freedom an picking the solution $\{a_j\}$ to (A.9) (and equivalently (A.10)). This has interesting interpretations in the context of the unidirectionality of causes and effects, See Louie et al. (1982), in particular, for details).

The metric geometry of our phenomenological calculus may be succinctly expressed in the following arrow diagram, which we call the *duality-invariance diagram* (DID):



(A.11)

With the phenomenological calculus we have a tool to study the natural world as manifested by phenomena: their genesis, their interrelationships, and in general, the morphology of representations of phenomena. The following four projection postulates (which were first proposed in Richardson and Louie (1983)) provide the link between the hidden and the observable:

Postulate P1. Nature is a Hilbert space over the real field, and it consists of invariants $\mathbf{I}$, which are not directly accessible.

Postulate P2. Real manifestations (i.e. observables, phenomena, or appearances) are projections of $\mathbf{I}$. (The term *projection* is taken here in its functional analytic sense, that it is an idempotent self-adjoint bounded linear operator. See Richardson and Louie (1983) for details.)

Postulate P3. It is possible to synthesize a representation, $\mathbf{R}$, of the relevant features of the invariant, $\mathbf{I}$, from measures on the appearances and their projectors. As far as reproducing the appearances, $\mathbf{R}$ is the best approximation to $\mathbf{I}$, in the sense that their metric distance $\| \mathbf{I} - \mathbf{R} \|$ is minimal.

Postulate P4. The representation $\mathbf{R}$ is the response tensor in our phenomenological calculus, and is invariant in form. In particular, there is a dual representation [which, when shown without the symbolism of projectors, is summarized by form (A.8) above].

Projection operators (except the trivial ones) are singular. This means that one can never completely reconstruct the original invariant $\mathbf{I}$ from the response tensors $\mathbf{R}$. Another way to state this is that we can never have $\| \mathbf{I} - \mathbf{R} \| = 0$, which is to say $\mathbf{I} = \mathbf{R}$, because a complete description of the invariant is something that we do not have enough information to achieve. This, of course, is an alternate characterization of complexity.

APPENDIX B. CARTESIAN TENSORS AND THEIR PRODUCTS

The base space under consideration is the Euclidean space $\mathbb{R}^n$ equipped with Cartesian coordinates. The summation convention is repeated (Roman) subscripts imply summation from l to n.

The (*standard*) *inner product* (or *dot product*) of two vectors (first-order tensors) **a** and **b** is a positive definite symmetric bilinear form, and may be defined component-wise as the sum

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i, \tag{B.1}$$

where $\mathbf{a} = a_i \mathbf{e}_i$ and $\mathbf{b} = b_i \mathbf{e}_i$ in some Cartesian basis $\{\mathbf{e}_i\}$. Note that the tensorial action of this inner product is to take two first-order tensors, reduce the total order by two, and hence yield a zeroth-order tensor (i.e. a scalar, a real number): $1 + 1 - 2 = 0$.

Now let **a** and **b** be fixed vectors, and allow a third vector **v** to vary. Define the symbol $\mathsf{T} \cdot \mathbf{v}$ by

$$\mathsf{T} \cdot \mathbf{v} = \mathbf{a}(\mathbf{b} \cdot \mathbf{v}). \tag{B.2}$$

The right-hand side of (B.2) is a vector multiplied by scalar, hence a vector; therefore so is the left-hand side. In fact, it is easily verified that $\mathsf{T} \cdot \mathbf{v}$ is a linear vector-valued function of the vector **v**. We may write T simply as the juxtaposition of **a** and **b**, $\mathsf{T} = \mathbf{ab}$, and rewrite (B.2) as

$$(\mathbf{ab}) \cdot \mathbf{v} = \mathbf{a}(\mathbf{b} \cdot \mathbf{v}). \tag{B.3}$$

We call **ab** a *dyad*, or *outer product* of the two vectors **a** and **b**.

The dyad is defined in (B.3) in terms of its action on the vector **v**, i.e. as a linear operator. We now need to invoke an important theorem in tensor algebra:

THEOREM. *For any two vector spaces X and Y, there is a natural isomorphism* $L(X,Y) \equiv X^* \otimes Y$, *i.e. between the space of linear transformations from X to Y and the space of bilinear functionals on* $X^* \times Y$, *where* X^* *is the dual vector space of X.*

One particular consequence of this is that the linear operator, the dyad **ab**, is equivalent to a bilinear form, i.e. a second-order tensor. It is trivial to verify that the components of the dyad $\mathsf{T} = \mathbf{ab}$ are

$$[\mathsf{T}]_{ij} = T_{ij} = [\mathbf{ab}]_{ij} = a_i b_j. \tag{B.4}$$

The outer product is non-commutative. Indeed, we have

$$\mathbf{ab} = (\mathbf{ba})^T, \tag{B.5}$$

where T^T is the *transpose* of the second-order tensor $\mathsf{T} = T_{ij}$, a second-order tensor defined by

$$\mathsf{T}^T = T_{ji}. \tag{B.6}$$

A second-order tensor T is called *symmetric* if $\mathsf{T} = \mathsf{T}^T$, whence $T_{ij} = T_{ji}$.

A finite sum of dyads is called a *dyadic*, which is clearly a second-order tensor. In Richardson et al. (1982), we showed that the set of all dyadics is in fact the space of all second-order tensors, when the base space is a Euclidean space. (And in Louie et al. (1982), the same is shown when the base space is a general Hilbert space.)

Note that (B.2) also defines the operation indicated by the "dot symbol" $\cdot$ in $\mathsf{T} \cdot \mathbf{v}$. This is the *inner product* of a second-order tensor and a vector, resulting in a vector. It retains the tensorial action of "reduce the total order by two": $2+1-2=1$. The components of this inner product $\mathsf{T} \cdot \mathbf{v}$ are

$$[\mathsf{T} \cdot \mathbf{v}]_i = T_{ij} v_j. \tag{B.7}$$

Unlike the inner product of two vectors, the inner product of a second-order tensor and a vector is non-commutative. In fact, the inner product of a vector and a second-order tensor, $\mathbf{v} \cdot \mathsf{T}$, needs a separate definition: for an arbitrary vector $\mathbf{w}$, let

$$(\mathbf{v} \cdot \mathsf{T}) \cdot \mathbf{w} = \mathbf{v} \cdot (\mathsf{T} \cdot \mathbf{w}). \tag{B.8}$$

The components of the vector $\mathbf{v} \cdot \mathsf{T}$ are

$$[\mathbf{v} \cdot \mathsf{T}]_i = v_j T_{ji}, \tag{B.9}$$

whence

$$\mathbf{v} \cdot \mathsf{T} = \mathsf{T}^T \cdot \mathbf{v}. \tag{B.10}$$

When T is a dyad ($\mathsf{T} = \mathbf{ab}$), a sequential application of the relations (B.10), (B.5), and (B.3) gives

$$\mathbf{v} \cdot (\mathbf{ab}) = (\mathbf{a} \cdot \mathbf{v})\mathbf{b}. \tag{B.11}$$

While the two inner products (B.7) and (B.9) are not equal, they are "dual" in the sense that they correspond to "column vector post-multiplication" and "row vector pre-multiplication" with a matrix, respectively, in linear algebra. One may choose one convention or another, and the resulting theories are equivalent.

Definition (B.8) also builds in an "associativity" property to the "triple product". We may therefore omit the parentheses, and write either side of the equality as $\mathbf{v} \cdot \mathbf{T} \cdot \mathbf{w}$. But note, however, that in general

$$\mathbf{v} \cdot \mathbf{T} \cdot \mathbf{w} \neq \mathbf{w} \cdot \mathbf{T} \cdot \mathbf{v}. \tag{B.12}$$

It is also readily verified that when $\mathbf{T}$ is a dyad, $\mathbf{T} = \mathbf{ab}$, we have

$$\mathbf{v} \cdot (\mathbf{ab}) \cdot \mathbf{w} = (\mathbf{a} \cdot \mathbf{v})(\mathbf{b} \cdot \mathbf{w}). \tag{B.13}$$

Proceeding hierarchically, we now let $\mathbf{S}$ and $\mathbf{T}$ be two second-order tensors and $\mathbf{v}$ be an arbitrary vector. The *inner product* of $\mathbf{S}$ and $\mathbf{T}$ is defined functionally by

$$(\mathbf{S} \cdot \mathbf{T}) \cdot \mathbf{v} = \mathbf{S} \cdot (\mathbf{T} \cdot \mathbf{v}). \tag{B.14}$$

The right-hand side of (B.14) is a vector, whence the inner product $\mathbf{S} \cdot \mathbf{T}$ is itself of a linear operator, thus a second-order tensor; the tensorial action is $2+2-2=2$. Its components are

$$[\mathbf{S} \cdot \mathbf{T}]_{ij} = S_{ik} T_{kj}. \tag{B.15}$$

This inner product, analogous to the standard multiplication of two matrices, is again not commutative, with

$$\mathbf{T} \cdot \mathbf{S} = (\mathbf{S}^T \cdot \mathbf{T}^T)^T. \tag{B.16}$$

When the second-order tensors are dyads, their inner product is also a dyad:

$$(\mathbf{ab}) \cdot (\mathbf{cd}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{ad} \tag{B.17}$$

(where the scalar $(\mathbf{b} \cdot \mathbf{c})$ may be considered a multiplier of either $\mathbf{a}$ or $\mathbf{d}$).

The three inner products (B.1), (B.7), and (B.15) are examples of a general tensorial process known as *contraction*. This is a process when pairs of indices of the tensors present are set equal, and the implied summation carried out. The number of "free" indices is then reduced by multiples of two, with a corresponding reduction in the sum of the tensorial orders. Contrariwise, the tensorial action of an outer product simply adds the orders.

Note that the three inner products (B.1), (B.7), and (B.15) yield tensors of order 0, 1 and 2, respectively. Since contraction reduces the total order by multiples of two, (B.15) is the only case in which further contraction is possible. Indeed, we now define the *double inner product* (or *Frobenius product*) of two second-order tensors $\mathbf{S}$ and $\mathbf{T}$ as

$$\mathbf{S} : \mathbf{T} = \mathrm{tr}(\mathbf{S} \cdot \mathbf{T}^T), \tag{B.18}$$

where $\mathrm{tr}(\mathbf{T})$ is the *trace* of the second-order tensor $\mathbf{T}$, a scalar defined by

$$\mathrm{tr}(\mathbf{T}) = T_{ii}. \tag{B.19}$$

The double inner product of two second-order tensors results in a scalar, with tensorial action $2+2-2 \times 2=0$. The value of $\mathbf{S} : \mathbf{T}$ in terms of components is

$$\mathbf{S} : \mathbf{T} = S_{ij} T_{ij}. \tag{B.20}$$

(Note that $\delta(\mathbf{S}, \mathbf{T}) = \mathbf{S} : \mathbf{T} = \mathrm{tr}(\mathbf{S} \cdot \mathbf{T}^T)$ is the dissipation metric tensor in Louie and Richardson (1986).) When $\mathbf{S}$ and $\mathbf{T}$ are dyads, $\mathbf{S} = \mathbf{ab}$ and $\mathbf{T} = \mathbf{cd}$, their double inner product is the scalar

$$(\mathbf{ab}) : (\mathbf{cd}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}). \tag{B.21}$$

Comparing this with (B.13), we see that

$$(\mathbf{ab}) : (\mathbf{cd}) = \mathbf{c} \cdot (\mathbf{ab}) \cdot \mathbf{d}. \tag{B.22}$$

Another "double contraction" of two second-order tensors is possible, by reversing the indices on one of the tensors; it may be represented as the double inner product

$$\mathbf{S} : \mathbf{T}^T = \mathbf{tr}(\mathbf{S} \cdot \mathbf{T}) = S_{ij} T_{ji}. \tag{B.23}$$

(Note that $\lambda(\mathbf{S}, \mathbf{T}) = \mathbf{S} : \mathbf{T}^T = \mathrm{tr}(\mathbf{S} \cdot \mathbf{T})$ is the Lorentz metric tensor in Louie and Richardson (1986).)

An *outer product* of a second-order tensor $\mathbf{T}$ and a vector $\mathbf{v}$ produces a third-order tensor $\underline{\mathbf{P}} = \mathbf{Tv}$ with components

$$[\underline{\mathbf{P}}]_{ijk} = P_{ijk} = [\mathbf{Tv}]_{ijk} = T_{ij} v_k. \tag{B.24}$$

When the second-order tensor involved is a dyad, the resulting outer product is called a *triad*, with

$$[\mathbf{abc}]_{ijk} = a_i b_j c_k. \tag{B.25}$$

A finite sum of triads is a third-order tensor called a *triadic*. The set of triadics is the space of all third-order tensors. (The same argument in Richardson et al. (1982) for dyadics and second-order tensors applies here.)

Both the (*single*) *inner product* and the *double inner product* between a second-order tensor S and a third-order tensor $\underline{\mathsf{P}}$ may be defined. They produce a third-order tensor $\underline{\mathsf{Q}} = \mathsf{S} \cdot \underline{\mathsf{P}}$ and a vector $\mathbf{a} = \mathsf{S} : \underline{\mathsf{P}}$, respectively, with components

$$Q_{ijk} = S_{il} P_{ljk}, \tag{B.26}$$

and

$$a_i = S_{jk} P_{jki}. \tag{B.27}$$

The tensorial actions are the contractions $2+3-2=3$ and $2+3-2\times 2=1$. When $\underline{\mathsf{P}}$ is the *outer product* of a second-order tensor T and a vector $\mathbf{v}$ ($\underline{\mathsf{P}} = \mathsf{T}\mathbf{v}$), we have

$$\mathsf{S} \cdot \mathsf{T}\mathbf{v} = (\mathsf{S} \cdot \mathsf{T})\mathbf{v}. \tag{B.28}$$

and

$$\mathsf{S} : \mathsf{T}\mathbf{v} = (\mathsf{S} : \mathsf{T})\mathbf{v}. \tag{B.29}$$

Both these inner products between S and $\underline{\mathsf{P}}$ are non-commutative, so that $\underline{\mathsf{Q}} = \underline{\mathsf{P}} \cdot \mathsf{S}$ is

$$Q_{ijk} = P_{ijl} S_{lk}, \tag{B.30}$$

and $\mathbf{a} = \underline{\mathsf{P}} : \mathsf{S}$ is

$$a_i = P_{ijk} S_{jk}. \tag{B.31}$$

The *triple inner product* of two third-order tensors $\underline{\mathsf{P}}$ and $\underline{\mathsf{Q}}$ produces a scalar $\underline{\mathsf{P}} \vdots \underline{\mathsf{Q}}$ $(3+3-3\times 2=0)$, which is the sum of n^3 products of the tensor components

$$\underline{\mathsf{P}} \vdots \underline{\mathsf{Q}} = P_{ijk} Q_{ijk}. \tag{B.32}$$

When $\underline{\mathsf{P}}$ and $\underline{\mathsf{Q}}$ are outer products of the form (B.24), the triple inner product is

$$\mathbf{Su} \,\vdots\, \mathbf{Tv} = (\mathbf{S} : \mathbf{T})(\mathbf{u} \cdot \mathbf{v}); \tag{B.33}$$

and when they are triads, the triple inner product reduces to the product of three inner products of vectors

$$\mathbf{abc} \,\vdots\, \mathbf{uvw} = (\mathbf{a} \cdot \mathbf{u})(\mathbf{b} \cdot \mathbf{v})(\mathbf{c} \cdot \mathbf{w}). \tag{B.34}$$

Note that among all the "inner products" defined in this appendix, only the standard inner product (B.1) of two vectors, the double inner product (B.20) of two second-order tensors, and the triple inner product (B.32) of two third-order tensors are inner products in the sense of "positive definite symmetric bilinear forms". The others are not symmetric, and are not even scalar-valued functions.

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