DUALITY AND INVARIANCE IN THE REPRESENTATION OF PHENOMENA

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Abstract—We present a mathematical formulation of invariance and duality, with special attention to their role in the biological sciences. By this means, a common morphology is illustrated with reference to such diverse dualities as Rosen's metabolic-repair systems [7], the force-flux conjugacy of irreversible thermodynamics, a cause-and-effect phenomenology for aging, and the recognition-response mechanism involved in enzyme catalysis, as substrate is transformed to product. It is demonstrated that invariance and duality are the fundamental concepts underlying the phenomenological calculus proposed by Richardson, Louie, and Swaminathan [9]. Furthermore, it is shown that this phenomenological approach is not limited to those systems which can be characterized by a response tensor, but can be generalized by use of a proposed duality-invariance diagram to include a much larger class of systems.

INTRODUCTION

The hypothesis of the universality of science presupposes the existence of invariants. In its most naive formulation, this principle of invariance insists upon the reproducibility of empirical results by different observers who faithfully copy a specified situation. From a more sophisticated viewpoint, universality is embedded in the proposition that all laws of nature are invariant under coordinate transformations. That is, the laws governing phenomena are independent of the observer. Conversely, alternate descriptions of the same phenomena are ultimately related by reference to whatever underlying invariances they have in common.

The role of mathematical modelling in science is more than merely to reproduce the phenomena. The goal must be to discover the interrelations between phenomena—to discover the morphology connecting the descriptions associated with a given invariant (or class of invariants). It is well-known in mathematics that in certain specified circumstances any given structure possesses a dual description and the morphology of the transformations within the general category of descriptions can be discovered in this duality.

"Any science is the study of a morphology" (René Thom [1]). Given a specific object, scientists of each discipline "observe" some class of phenomena they deem worthy of study. The resultant phenomenology invariably turns out to be a spatial/temporal/organizational

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morphology. This paper is on such a morphology. The morphology associated with the concepts of invariance and duality is formulated in precise mathematical terms, and diverse examples are presented to demonstrate its central role in science.

1. INVARIANCE

Lao Tse, in *Tao Te Ching*, said that "The essence of life is change". If one accepts the idea that every change is in principle observable (Rosen [2] and Louie [3]) and the idea that science has to do with the description of reality, then one might conclude that the ultimate science would consist of a tremendous catalogue of all events in all parts of space-time. Surely such a cosmic catalogue of facts is not science. Given these data, a selection must be made and systematic organization provided. The job of the scientist is, in effect, to create or to recognize—"order out of chaos". And to this end, theorization is the fundamental tool. Theorization, again to quote Thom [4], is "to recognize the regularities among the spatiotemporal appearances, patterns or structures", and then "to express these either in terms of the reproducibility of phenomena, or by equivalence classes between appearances".

The basic objective of theory in science is, then, the reduction of arbitrariness through unifying principles. We wish to suggest that "invariance of form" is such a unifying principle—a principle of such generality that it might aptly be designated a principle of natural philosophy.

The idea of invariance in mathematics is perhaps best illustrated by tensors (Weatherburn [5]). A tensor $\underline{\underline{T}}$ of type (r, s) over a real vector space V is recognized as such if and only if, given any two sets of basis vectors $\{\underline{e}_j\}$, $\{\underline{e}^i\}$ for V and V^*, respectively, there are real numbers (the components of $\underline{\underline{T}}$ with respect to these bases) $T^{a_1 \cdots a_r}_{b_1 \cdots b_s}$ such that $\underline{\underline{T}}$ has the form

$$\underline{\underline{T}} = T^{a_1 \cdots a_r}_{b_1 \cdots b_s} \underline{e}_{a_1} \otimes \cdots \otimes \underline{e}_{a_r} \otimes \underline{e}^{b_1} \otimes \cdots \otimes \underline{e}^{b_s} \tag{1}$$

(the Einstein convention of summing over pairs of repeated upper and lower indices will be used throughout this paper); *and* when under a change of bases $e_{j'} = A^j_{j'}\underline{e}_j$, $e^{i'} = B^{i'}_i\underline{e}^i$, the equations

$$T^{a'_1 \cdots a'_r}_{b'_1 \cdots b'_s} = B^{a'_1}_{a_1} \cdots B^{a'_r}_{a_r} A^{b_1}_{b'_1} \cdots A^{b_s}_{b'_s} T^{a_1 \cdots a_r}_{b_1 \cdots b_s} \tag{2}$$

hold. In other words, a tensor is *invariant* in its form (1) under coordinate transformations.

The absolute calculus for tensors, developed as pure mathematics by Ricci and Levi-Civita at the turn of this century, was recognized by the physicist Einstein as the natural tool for formulating gravitational field equations. The cornerstone of his theory is the Principle of General Covariance: the field equations must be expressed in covariant* tensor form. The genius of Einstein (and his mathematician colleagues) was to show that the physics of gravitation was reducible to the morphology of spacetime—a morphology based upon the postulated invariance of a Riemannian line element. Even outside the realm of general relativity, if science is to be independent of the particular vantage point of a given observer, all laws of nature must be expressible in covariant* forms. There is no need for us to discuss in detail the well-known role of tensors and their invariant properties in relativity theory. However, this specific example by no means sets the limits to their applicability, and we shall return to tensors later in the paper.

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*The word *covariance* is commonly used in two different circumstances. It is used here in the sense of being invariant in form under coordinate transformations. The more restricted meaning of being dual to contravariant is used in the remainder of the paper.

An excellent example of invariance in the natural sciences is the relational biology of Rashevsky [6]. It was his idea that organisms are recognized as such because we can observe homologies in their behaviors, regardless of the many different physical structures through which these behaviors are exhibited. Thus, all organisms manifest the same set of basic and ubiquitous biological functions, and through this manifestation, organisms can be mapped on one another in such a way as to preserve these basic relations. This recognition of the *invariance of functional form* led to the formulation of Rashevsky's principle of biotopological mappings and, consequently, to the remarkable field of relational biology. An interesting relational treatment of biology is given by Rosen's [7] (M, R)-systems. These systems will be discussed in Sec. 6.

2. DUALITY

The dictionary definition of duality is "twofold condition". Underlying this is the essence of being "plural", the importance of "alternate descriptions" as discussed in Rosen [2] and Louie [3].

In the language of mathematics, we shall make the following definition.

Duality is a functor (covariant or contravariant) D from a category $\mathfrak{A}$ to itself such that D^2 is naturally equivalent to the identity functor $I_{\mathfrak{A}}$ on $\mathfrak{A}$ (in particular, for each $\mathfrak{A}$-object X, D^2X is $\mathfrak{A}$-isomorphic to X).

Note that by definition, D is necessarily an isomorphism as a functor. Sometimes we will relax the condition and only consider $D: \mathfrak{A} \to \mathfrak{A}$ on the $\mathfrak{A}$-objects, not requiring D to preserve morphisms, such as when we discuss the various principles of duality.

The subject matter is best introduced by an example, the relation between a real, finite-dimensional vector space V and its *dual* space V^* of all linear functionals on V. V^* is also a real vector space of the same dimension as V (and the two spaces are isomorphic). Given a basis $\{\underline{e}_j : j = 1, 2, \ldots, n\}$ for V, there is correspondingly a *dual* basis $\{\underline{e}^i : i = 1, 2, \ldots, n\}$ for V^* with the property that $\underline{e}^i(\underline{e}_j) = \delta^i_j$, the Kronecker delta. Any n-dimensional vector $\underline{x}$ has a dual representation $x_i\underline{e}^i = \underline{x} = x^j\underline{e}_j$ for unique sets of real numbers $\{x_i\}$ and $\{x^j\}$. The representation of $\underline{x}$ is invariant in the sense that it takes the same form (morphology) in V^* and V, namely, a real linear combination of n basis vectors.

If we iterate the above process, we obtain the dual V^{**} of all linear functionals on V^*, alias the "second dual space" of V, which is *naturally* isomorphic to V. The isomorphism $V \cong V^{**}$ is $\underline{x} \mapsto \hat{\underline{x}}$ defined by $\hat{\underline{x}}(y) = y(\underline{x})$ for $y \in V^*$. Further, the basis for V^{**} which is dual to the dual basis $\{\underline{e}^i\}$ for V^* is the set $\{\underline{e}_j\}$ itself, after the identification $\underline{x} = \hat{\underline{x}}$ is made. (That the isomorphism $V \cong V^{**}$ is termed *natural*, but not the isomorphism $V \cong V^*$, is precisely due to this fact: that the former is independent of the choice of the basis while the latter is not.)

This example is on the category $\mathfrak{A} = \mathfrak{Vect}$, that of vector spaces and linear transformations. The duality functor is $D = (\ )^*$, sending a vector space V to its dual $D(V) = V^*$. And we have $D^2(V) \cong V$. For a linear transformation $T: V \to W$, $D(T) = T^*: W^* \to V^*$ is the "adjoint" of T defined by $T^*(y) = y \circ T \in V^*$ for each $y \in W^*$. And again $T^{**} = T$. One easily checks that D is a contravariant functor from $\mathfrak{Vect}$ to $\mathfrak{Vect}$, satisfying $D^2 = I$. Thus, D is indeed a duality by our definition.

We shall use also the symbol D to denote the map which sends $\{\underline{e}_j\}$ to $\{\underline{e}^i\}$, and again use the same symbol D to denote the map $\{x_i\} \mapsto \{x^j\}$ (here D is simply the metric tensor $g = (g^{ij}) = (\underline{e}^i \cdot \underline{e}^j)$). Hence, D is really a map at three hierarchical levels:

$$D : \begin{cases} V^* \leftarrowtail V \\ \{\underline{e}^i\} \leftarrowtail \{\underline{e}_j\} \\ \{x_i\} \mapsto \{x^j\} \end{cases} \tag{3}$$

The different hierarchical level maps may be safely denoted by the same letter D because we can always tell from the arguments which level is meant.

The content of this example in linear algebra can be summarized in the following diagram:

$$
\begin{array}{ccc}
 & D & \\
x_i \xrightarrow{\hspace{3cm}} & & x^j \\
\diagdown & x_i \underline{e}^i = \underline{x} = x^j \underline{e}_j & \diagup \\
\underline{e}^i \xleftarrow{\hspace{3cm}} & & \underline{e}_j \\
 & D &
\end{array}
\tag{4}
$$

Note that the middle line of diagram (4) contains the invariance principle $x_i \underline{e}^i = x^j \underline{e}_j$ implied by the duality $D: V^* \leftrightarrow V$.

Such a consideration of vector spaces and their duals is but one example of many similar mathematical and natural situations. It appears that in almost every branch of mathematics there is a duality principle. In many instances, duality implies invariance, which is central to the development of the subject. In order to deal in a general way with such situations, we shall introduce the concept of a *duality-invariance diagram* (DID, for short) for which the above diagram (4) is a specific example. The DID is the morphology to be studied in this paper, and it will take the following form:

$$
\begin{array}{ccc}
 & D & \\
x \xrightarrow{\hspace{3cm}} & & y \\
\diagdown & ax = by & \diagup \\
a \xleftarrow{\hspace{3cm}} & & b \\
 & D &
\end{array}
\qquad
\begin{array}{c}
(5) \\
(6) \\
(7)
\end{array}
$$

The two mappings indicated in lines (5) and (7) form the "duality" portion of the DID, while the equality in line (6) is the "invariance" portion. After we consider some more examples of duality in the next section, the concept of the DID will be illustrated by the phenomenological calculus of Richardson [8], Richardson *et al.* [9], and Louie *et al.* [10], with realizations of the natural processes of aging, causality, recognition, transport, etc., presented therein.

3. EXAMPLES OF DUALITY IN MATHEMATICS

Let $\mathfrak{A} = \mathfrak{Cat}$, the category of all (small) categories and functors, and let $D = (\ )^{op}$ send a category $\mathfrak{C}$ to its opposite $D\mathfrak{C} = \mathfrak{C}^{op}$. A functor $F: \mathfrak{B} \to \mathfrak{C}$ is mapped to the functor DF: $\mathfrak{B}^{op} \to \mathfrak{C}^{op}$ defined by $DF(\mathfrak{B}^{op}\text{-object } X) = FX$ and $DF(f^{op} \in \mathfrak{B}^{op}(Y, X)) = (Ff)^{op} \in \mathfrak{C}^{op}(FY, FX)$. $D = (\ )^{op}$ is then a covariant functor, and satisfies the duality requirement $D^2 = I$.

Let us now relax the restrictions on a duality D and consider it simply as a function from a set A to itself, such that D^2 is the identity function I_A on A. In this setting we shall look at the various principles of duality.

First, let A be the algebraic theory of categories, i.e. let A be the set of all logical statements in category theory. Let $D: A \to A$ send a statement Σ in A to the statement $D\Sigma$ defined by $D\Sigma(\mathfrak{C}) = \Sigma(\mathfrak{C}^{op})$ for every category $\mathfrak{C}$; i.e., D replaces every categorical concept in Σ by its corresponding "co-concept". An example is in order. Suppose $\Sigma(\mathfrak{C})$ is the statement, "Products in a category $\mathfrak{C}$ are unique up to isomorphism", then $D\Sigma(\mathfrak{C}) = \Sigma(\mathfrak{C}^{op})$ is the statement, "Products in a category $\mathfrak{C}^{op}$ are unique up to isomorphism", which is the same as, "Coproducts in a category $\mathfrak{C}$ are unique up to isomorphism". (Note that "co-isomorphisms" are isomorphisms. Those readers who are not familiar with category theory are referred to the introductory text, Arbib and Manes [11]). Clearly $D^2 = I$. The principle of categorical duality is this: Σ is true if and only if $D\Sigma$ is true. For example, the above

statement about uniqueness of products is true (i.e., it is a consequence of the definitions and axioms of category theory). Once that is established, the truth of the statement about uniqueness of coproducts follows immediately, without requiring an independent proof. Thus, duality cuts the work in half.

Next, let A be the theory of Boolean algebras. A Boolean algebra is represented by $B = \langle B, \wedge, \vee, *, 0, 1 \rangle$ where the components denote the set, infimum, supremum, complement, least element, and greatest element, respectively. (A good introduction to Boolean algebras is Halmos[12]). Let $D: A \rightarrow A$ be defined by sending a statement Σ in A to the dual statement $D\Sigma$, obtained by interchanging $\wedge$ with $\vee$, and 0 with 1. For example, if Σ is the statement "$x \vee x^* = 1$", then $D\Sigma$ is "$x \wedge x^* = 0$". Again it is clear that $D^2 = I$. The principle of duality for Boolean algebras is that, if Σ holds in all Boolean algebras, then so does its dual. The prototype of Boolean duality is that in set theory—the power set PX of a set X is a Boolean algebra $\langle PX, \cap, \cup, {}^c, \phi, X \rangle$. For example, once the truth of one of de Morgan's laws, say "$(A \cap B)^c = A^c \cup B^c$", is established, the other, "$(A \cup B)^c = A^c \cap B^c$", follows automatically.

We shall consider another one of the many more duality principles as our final example. Let us put $A = $ the theory of plane projective geometry. (See Ayres [13] for a survey of projective geometry.) D shall send a statement Σ in A to its dual $D\Sigma$ obtained from Σ by interchanging "point" with "line". Say $\Sigma = $ "Any two distinct points determine a unique line", then $D\Sigma = $ "Any two distinct lines determine a unique point". The principle of duality for plane projective geometry is then: Σ is true if and only if $D\Sigma$ is true.

4. RESPONSE TENSOR AND DID

A good illustration of the concepts developed so far is the phenomenological calculus from [8–10]. This calculus, being the "bilinear algebra" of tensors of type (1, 1), is a natural extension of the linear algebra example of the DID (4). Here the domain is the space $T_1^1(H)$ of dyadics, where $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$ is a real Hilbert space. The map $(L^{ij}) = (\langle a^i, a^j \rangle)$, where $a^i \in H^*$, $i = 1, 2, \ldots, m$, sends $F_i \in H$ to $J^j = L^{ij}F_i \in H^*$, and sends $a_j \in H$ back to $a^i = L^{ij}a_j$. (See [10] for details.) We then have a dyadic of the form $a^iF_i \in T_1^1(H)$ and a dyadic of the form $a_jJ^j \in T_1^1(H^*)$. A consequence of the linear algebra DID (4) is that $H \cong H^{**}$, whence $T_1^1(H) \cong (T_1^1(H^*))^* \cong T_1^1(H^{**})$. We can then formulate the duality D as the mapping sending $T_1^1(H)$ to $T_1^1(H^*)$, and the *principle of dyadic invariance* as $a^iF_i = a_jJ^j$, and give this common object the name of a *response tensor*, $\underline{R}$.

Analogously to diagram (3), we could use the same symbol D to denote the map at three hierarchical levels:

$$D: \begin{cases} T_1^1(H) \mapsto T_1^1(H^*) \\ \{a^i\} \leftarrowtail \{a_j\} \\ \{F_i\} \mapsto \{J^j\}, \end{cases} \tag{8}$$

and the corresponding DID is

$$ \begin{array}{c} F_i \xrightarrow{\quad D \quad} J^j \\ \diagdown a^iF_i = \underline{R} = a_jJ^j \diagdown \\ a^i \xleftarrow{\quad D \quad} a_j. \end{array} \tag{9}$$

Although diagrams (3)–(4) and (8)–(9) have the same morphologies, there is a fundamental

difference between them. The vector space duality D in diagram (3) is a duality (in the sense that $D^2 = I$) at all three hierarchical levels. Explicitly, applying D twice would yield the following:

$$D^2: \begin{cases} V = V^{**} \xleftarrow{\;D = (\ )^*\;} V^* \xleftarrow{\;D = (\ )^*\;} V \\ {} \\ \{\underline{e}_j\} \xleftarrow[D:\ \epsilon_j(e^i) = \delta_j^i]{D = (e_i \cdot e_j) = (g_{ij})} \{\underline{e}^i\} \xleftarrow[D:\ \underline{e}^i(e_j) = \delta_j^i]{D = (e^i \cdot e^j) = (g^{ij})} \{\underline{e}_j\} \\ {} \\ \{x^j\} \xrightarrow[D = (e_i \cdot e_j) = (g_{ij})]{} \{x_i\} \xrightarrow[D = (e^i \cdot e^j) = (g^{ij})]{} \{x^j\}. \end{cases} \qquad (10)$$

That $D^2 = I$ on the coordinates and the components is a consequence of the metric tensor (g^{ij}) being invertible with $(g^{ij})^{-1} = (g_{ij})$. Contrariwise, the dyadic duality D in diagram (8) is only a true duality at the level of the tensor spaces:

$$D^2: T_1^1(H) \xmapsto{D} T_1^1(H^*) \xmapsto{D} T_1^1(H^{**}) \cong T_1^1(H). \qquad (11)$$

At both the levels of the coordinates and the components, the map

$$D = (\langle a^i, a^j \rangle) = (L^{ij}): \begin{cases} \{a^i\} \leftharpoonup \{a_j\} \\ {} \\ \{F_i\} \mapsto \{J^j\} \end{cases} \qquad (12)$$

is *not* invertible and there is no unique map sending $\{a^i\}$ back to $\{a_j\}$ and $\{J^j\}$ back to $\{F_i\}$. Since $\{F_i\}$ is interpreted as the set of causes and $\{J^j\}$ as the set of effects, we obtain the statement of the unidirectionality of causality: that causes imply effects but not vice versa. Thus, the non invertibility of the matrix of phenomenological coefficients (L^{ij}) is an asset, rather than a hindrance, to the development. This situation is discussed in detail in [9].

5. TENSOR ALGEBRA

In this section we would like to consider a variation of the DID, a commutative diagram depicting duality and invariance properties unique to tensors (and thus to those natural systems represented by tensors). Duality D for tensors follows from that between a vector space and its dual, and invariance A for tensors (recalling from Sec. 1) is that under a coordinate transformation. These are represented succinctly in

$$\begin{array}{ccc} T_s^r(V) & \xrightarrow{\;D\;} & T_r^s(V) \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle D(A)} \\ T_s^r(V) & \xrightarrow[\;D\;]{} & T_r^s(V) \end{array} \qquad (13)$$

Let us once again use the linear algebra example. The contravariant (type $(1, 0)$) representation of a vector $\underline{x}$ is $x^j \underline{e}_j$ for a basis $\{\underline{e}_j\}$ of $V = T_0^1(V)$. The covariant (type $(0, 1)$) representation is $x_i \underline{e}^i$ where $\{\underline{e}^i\}$ is the corresponding dual basis in $V^* = T_1^0(V)$. We shall pass the duality map $D: V \mapsto V^*$ down the hierarchy to the elements and denote $D: x^j \underline{e}_j \mapsto x_i \underline{e}^i$. This latter D map is, of course, really the pair of maps $((g_{ij}), (g^{ij}))$.

If $\{\underline{e}_{j'}\}$ is another basis of V, and $\underline{e}_{j'} = A_{j'}^j \underline{e}_j$, then $\underline{x} = x^j \underline{e}_j = x^{j'} \underline{e}_{j'} = x^{j'} A_{j'}^j \underline{e}_j$ whence $x^j = x^{j'} A_{j'}^j$. Let us denote this change-of-coordinate transformation by $A = ((A^{-1})_{j'}^j, A_{j'}^j): x^j \underline{e}_j \mapsto x^{j'} \underline{e}_{j'}$. On the other hand, if $\{\underline{e}^{i'}\}$ is dual to $\{\underline{e}_{j'}\}$, then $\underline{e}^i = \underline{e}^{i'} A_{i'}^i$ and $\underline{x} = x_i \underline{e}^i = x_{i'} \underline{e}^{i'}$ give $x_{i'} = A_{i'}^i x_i$. We

shall denote this $D(A) = (A^i_{i'}, (A^{-1})^{i'}_i)$: $x_i \varrho^i \mapsto x_{i'} \varrho^{i'}$. We then have

$$
\begin{array}{ccc}
x^j \varrho_j & \xrightarrow{\ D\ } & x_i \varrho^i \\
A \downarrow & & \downarrow {\scriptstyle D(A)} \\
x^{j'} \varrho_{j'} & \xrightarrow[\ D\ ]{} & x_{i'} \varrho^{i'}
\end{array}
\tag{14}
$$

which is the element-chasing version of

$$
\begin{array}{ccc}
T^1_0(V) & \xrightarrow{\ D\ } & T^0_1(V) \\
A \downarrow & & \downarrow {\scriptstyle D(A)} \\
T^1_0(V) & \xrightarrow[\ D\ ]{} & T^0_1(V)
\end{array}
\tag{15}
$$

Analogously, our dyadic response tensor $\underline{R}$ of type $(1, 1)$ has a diagram of the same kind as (14). A coordinate transformation $A\colon a^i F_i \mapsto a^{i'} F_{i'}$ gives rise to

$$
\begin{array}{ccc}
a^i F_i & \xrightarrow{\ D\ } & a_j J^j \\
A \downarrow & & \downarrow {\scriptstyle D(A)} \\
a^{i'} F_{i'} & \xrightarrow[\ D\ ]{} & a_{j'} J^{j'}
\end{array}
\tag{16}
$$

6. DID REPRESENTATION OF (M, R)-SYSTEMS

(M, R)-systems were created by Rosen [7] as a class of metaphorical, relational paradigms for cellular activities. The basic concepts underlying the (M, R)-systems were an outgrowth of the observation that the activities of all cells, however diverse, could generally be classified into two types, *metabolic* and *repair* (genetic). The class of relational models which arose when this intuition was abstractly formulated are the so-called (M, R)-systems. Such a system contains an array of interconnected components, $M_1, M_2, \ldots, M_n$, playing the role of the "metabolic" part. To each component M_i there is associated a system R_i, which accepts as inputs a certain subset of the outputs of the metabolic system, and produces as outputs new copies of the associated component M_i. The R_i's form the "repair" portion. These ideas are most appropriately formulated in the language of category theory. Each metabolic component M_i is represented by a mapping $f_i\colon A_i \to B_i$, where A_i and B_i represent the sets of inputs and outputs to the component, respectively. In category theory notation, this is $f_i \in \mathfrak{Ens}(A_i, B_i)$, $\mathfrak{Ens}$ being the category of (small) sets and functions. A repair component R_i is represented by a mapping Φ_i whose domain is a cartesian product of output sets B_j, and whose codomain is the set of morphisms $\mathfrak{Ens}(A_i, B_i)$. The simplest (M, R)-system could thus be represented by the diagram

$$
A \xrightarrow{\ f\ } B \xrightarrow{\ \Phi\ } \mathfrak{Ens}(A, B)
\tag{17}
$$

and it was shown (Rosen [14]) that in principle every abstract (M, R)-system is reducible to this simple form by making A, B, and f sufficiently complex. Henceforth we shall concentrate on this form.

Before we can establish the connection between (M, R)-systems and DIDs, we need the

following results from category theory. Let X be an object in an arbitrary category $\mathfrak{C}$. Then a natural "dual object" of X is the covariant hom-functor $\mathfrak{C}(X, \cdot)$. Note that $\mathfrak{C}(X, \cdot)$ sends a $\mathfrak{C}$-object Y to the set $\mathfrak{C}(X, Y)$, which is an object in the "base category" $\mathfrak{Ens}$. This is an interesting generalization of the linear algebra example, in which the dual object of a vector $\underline{x}$ is a linear functional $\underline{x}^*$ which sends a vector $\underline{y}$ to a real number $\underline{x}^*(\underline{y})$, which is an object in the "base field" $\mathbb{R}$. Capturing this idea, the "second dual" $\hat{Y}$ of Y (analogous to $\hat{y}$ in the linear algebra example) is the "evaluation map" which sends a dual object $\mathfrak{C}(X, \cdot)$ to the set $\mathfrak{C}(X, Y)$. We then have set up an isomorphism between $\mathfrak{C}$-objects and their second duals, but this says that $X \mapsto \mathfrak{C}(X, \cdot)$ is like a duality. The dual map $\hat{Y}: \mathfrak{C}(X, \cdot) \mapsto \mathfrak{C}(X, Y)$ is equally well represented by the corresponding contravariant hom-functor $\mathfrak{C}(\cdot, Y): X \mapsto \mathfrak{C}(X, Y)$, and the DID is

$$
\begin{array}{ccc}
X & \longrightarrow & \mathfrak{C}(X, \cdot) \\
& \mathfrak{C}(X, Y) & \\
\mathfrak{C}(\cdot, Y) & \longleftarrow & Y
\end{array}
\tag{18}
$$

Finally, applying this to diagram (17), in which $\mathfrak{C} = \mathfrak{Ens}$, we obtain the DID representation of the (M, R)-system

$$
\begin{array}{ccc}
A & \longrightarrow & \mathfrak{Ens}(A, \cdot) \\
& \mathfrak{Ens}(A, B) & \\
\mathfrak{Ens}(\cdot, B) & \longleftarrow & B
\end{array}
\tag{19}
$$

The covariant hom-functor $\mathfrak{Ens}(A, \cdot)$ has the interesting interpretation that it is the first half of the repair machinery determined (anticipatorily) by the inputs A of the metabolic system, while the contravariant hom-functor $\mathfrak{Ens}(\cdot, B)$ can depict the second half determined (retrospectively) by the outputs B. And the two halves together determine the invariant portion of the diagram, the (genetic) repair system.

7. ADJOINTNESS

The reader may have noticed that in the last section we stated that "$X \mapsto \mathfrak{C}(X, \cdot)$ is *like* a duality", rather than that it *is*. This is because in the definition of duality, D is required to map a category $\mathfrak{A}$ to *itself*; but $X \mapsto \mathfrak{C}(X, \cdot)$ sends $\mathfrak{C}$ to $\mathfrak{Ens}^{\mathfrak{C}}$, and it just so happens that there is a "natural dual" mapping in the other direction, and the composition of the two gives the isomorphism between $\mathfrak{C}$-objects and their second duals. This is the property we shall address in this section; we shall call it *adjointness*.

The notion of an adjoint operator is an old one, appearing for differential equations in the work of Legendre. In linear algebra, adjoints appear as conjugate transpose operators (as exemplified in Sec. 2). For a Hilbert space H, the adjoint T^* of a linear operator T on H is defined by

$$
\langle x, Ty \rangle = \langle T^*x, y \rangle
\tag{20}
$$

for all $x, y \in H$. The definition of *adjoint functor*, the generalization of all of the above, was first stated by Kan [15]. For functors $G: \mathfrak{B} \to \mathfrak{X}$ and $F: \mathfrak{X} \to \mathfrak{B}$, F is a *left adjoint* of G (and G is a *right adjoint* of F) if there is an isomorphism

$$
\mathfrak{X}(X, GB) \cong \mathfrak{B}(FX, B)
\tag{21}
$$

of hom sets, defined for all $\mathfrak{X}$-objects X and all $\mathfrak{B}$-objects B, and natural in these objects. Note the analogy of (21) to (20).

For example, if $\mathfrak{B} = \mathfrak{Vect}$ and $\mathfrak{X} = \mathfrak{Ens}$, and G is the functor which assigns to each real vector space B its underlying set of vectors (i.e., G is the "forgetful" functor, which forgets the vector space structure), then the corresponding left adjoint is the functor F which assigns to each set X the vector space FX with basis X. The isomorphism (21) is then the familiar one, which states that a linear transformation $T: FX \rightarrow B$ on the vector space FX with basis X is completely determined once its values $T': X \rightarrow GB$ on the basis X are known.

Many other examples of adjoint functors can be found in any one of the standard sources (e.g., Mac Lane [16]). For our purpose, note that duality is a left (and right) adjoint of itself. For $D: \mathfrak{A} \rightarrow \mathfrak{A}$, the isomorphism of hom sets is

$$\mathfrak{A}(X, DY) \cong \mathfrak{A}(DX, Y) \tag{22}$$

given by $f \in \mathfrak{A}(X, DY) \rightarrow D(f) \in \mathfrak{A}(DX, D^2Y) \cong \mathfrak{A}(DX, Y)$, by the very definition of a functor, and that $D^2 = I$ implies D is an isomorphism as a functor.

Further, in (21), if we put $B = FX$, we have

$$\mathfrak{X}(X, GFX) \cong \mathfrak{B}(FX, FX) \tag{23}$$

and the identity morphism $1_{FX} \in \mathfrak{B}(FX, FX)$ would be naturally isomorphic to a special morphism $X \rightarrow GFX$. Thus, adjoint functor is a natural extension of duality in that instead of $X \cong D^2X$ we have now a comparable condition "$X \cong GFX$". Roughly speaking, instead of "do something twice and get back where one started" as in a duality, adjointness is the condition "perform a twin pair of operations and get back where one started".

The concept of adjoint functors is summarized in the following diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\quad F \quad} & FX \\
& \searrow \mathfrak{X}(X, GB) \cong \mathfrak{B}(FX, B) \swarrow & \\
GB & \xleftarrow{\quad G \quad} & B
\end{array}
\tag{24}
$$

Duality provides a natural, or one might even say a canonical, pair of alternate descriptions in that it provides direct mathematical access to the morphology of the relationships between phenomena. The mathematical representation of the dualities presented above and/or in the references accord with observation and intuition: for example, force-and-flux, cause-and-effect, metabolism-and-repair, recognition-and-response as in enzyme-mediated catalysis, etc. But by the very nature of the components of the dual pair—that they signify plurality and denote alternate descriptions—they admit different characterizations as objects from different categories. The examples, therefore, have necessitated the transition to the more general concept of adjointness where the symmetric cycle D^2 around the invariance in the DID (5) is replaced by the asymmetric cycle GF in (24). This, for one thing, allows a precise characterization of the directionality (irreversibility or noninvertibility) of "twofold conditions" inherent in natural systems.

8. PROSPECTS: ADJOINT-INVARIANCE DIAGRAM

We end this essay on natural philisophy with the hope that the rather powerful mathematical formalism developed above has provided, beyond the indicated practical applications, a partial answer to a basic epistemological problem. How do we, as humans,

perceive the natural world? What constant (invariant) features underlie the appearances (morphologies, forms, and phenomena) so that there are regularity, reproducibility, and unification of perceptual experience? The classification and study of these invariances is the role of theory in science. Although we examined the morphology implied by invariance and duality, we did not discuss explicitly in what manner the invariances are made manifest in the phenomenological world as observables or how one might synthesize a representation of the invariant, given measures of the observables.

The concept of phenomena as projections of ideal, invariant objects is probably as old as philosophy itself and, stripped of its metaphysical overtones, remains even today a powerful metaphor in science. An observation (measurement of phenomena) as a projection upon a meter of some universal state space is not, however, a sufficient primary set. There are invariants, projections, and also projectors! For example, the components $\{x^i\}$ of a vector $\underline{x}$ assume meaning as projections only when considered in relation to the basis $\{\underline{e}^i\}$ as projectors: i.e. $x^i = \underline{e}^i(\underline{x})$. Furthermore, it requires both the projections and the (adjoint) projectors to synthesize a representation of the vector as $\underline{x} \equiv x^i\underline{e}_i$. Thus, a representation of the relevant features of the invariants can be synthesized only when we have some measures of both the projections and the projectors. This is exemplified not only by the representation of a vector by its components and its basis set, but also by that of a repsonse tensor by the causes and the constitutive parameters, and of an abstract cell by the metabolic and the repair portions, just to name a few invariants considered in this paper. Ultimately, our phenomenological study of morphology is based on the projection-projector couple, or more generally, an adjoint pair of functors. Diagram (24) of adjoint functors immediately suggests the following generalization of the DID:

$$
\begin{array}{ccc}
x & \xrightarrow{\;\;F\;\;} & y \\
 & ax \cong by & \\
a & \xleftarrow{\;\;\;\;\;\;} & b \\
 & G &
\end{array}
\tag{25}
$$

$F: x \mapsto y$ and $G: b \mapsto a$ form a pair of adjoints, and the middle line is an invariance of form, a natural isomorphism of structures. Diagram (25) will be called an *adjoint-invariance diagram* (AID for short). The morphology embedded in the AID will be the subject of the next paper in our exploration of the fundamentals of measurement and representation of natural systems.

In this paper we have made little distinction between the structures of mathematical objects and the structures of representations of natural objects, passing freely from one to the other in the text. Of this remarkable homology between mathematics and the existent world, E. P. Wigner [17] says:

The language of mathematics reveals itself unreasonably effective in the natural sciences ... a wondeful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.

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