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IRREVERSIBLE THERMODYNAMICS, QUANTUM MECHANICS AND INTRINSIC TIME SCALES

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Abstract—It is often assumed that phenomenology is a rather weak tool for the analysis of natural systems because it lacks generality. However, in a series of papers we have developed a phenomenological calculus based upon a general theory of measurement and mathematical representations (or, equivalently, upon system response as a bilinear form) which has a broad range of application. The present paper illustrates its power and versatility by demonstrating that irreversible thermodynamics and quantum mechanics are homomorphic. This result is, in itself, interesting since it shows that a large class of dissipative, deterministic systems are homomorphic to a large class of ideal, stochastic systems. In both cases, the metrical structure of the phenomenological calculus allows us to define a "proper time" intrinsic to the system dynamics. With this intrinsic time, a dynamics of aging can be defined upon the system's parameter space. In this context, Schrödinger's equation is seen as a dynamics of aging.

1. INTRODUCTION

The primary purpose of this paper is to demonstrate that irreversible thermodynamics and quantum mechanics are similar in their mathematical structure: that is, they are homomorphic. The mathematical structure of quantum mechanics has long been recognised[1, 2], and it can be said with much justification that the study of the physics of quantum mechanics is the study of this structure. In fact, a disconcerting lack of physical explanation for the axioms which define this structure (and which have proved to be astoundingly successful in their practical application) played an important part, along with relativity, in the revolution in viewpoint leading to modern, structurally oriented physics. On the other hand, irreversible thermodynamics[3] is a complex mixture of continuum mechanics and Gibbsian thermostatics with an arcane notation which in the end is rendered down to a relatively simple and useful linear formalism. Aside from linearity, there is no mathematical structure, the remainder of the formalism actually being physical constraints. The quadratic form representing dissipation is positive definite by virtue of the Second Law, and the symmetry of the coupling coefficients is given by an extrapolation to macroscopic flows of Onsager's investigation of relaxations in microscopic systems. Truesdell[4] offers a trenchant critique of the classical presentation of irreversible thermodynamics. However, in the present paper we are not interested in probing the foundations of irreversible thermodynamics or in seeking the limits to its realm of validity. Rather we accept the linear formalism, abstract it from its usual physical–chemical context, and show that it has an elegant metric structure—a structure homomorphic to that of quantum mechanics.

A secondary topic in this paper is a canonical definition of an intrinsic time scale based upon the norm associated with the metrical structures alluded to above. This intrinsic time is analogous to the proper time of relativity, which is based upon the norm associated with the Minkowski metric of four-space. The concept of intrinsic time provides a natural way to frame a dynamics of aging in the constitutive parameter space of dissipative systems. By means of the demonstrated homomorphism, these ideas can be immediately extended to quantum mechanical systems, where the Schrödinger equation can be interpreted as a dynamics of aging.

Irreversible thermodynamics and quantum mechanics, in their mathematical structure, are special cases of a general phenomenological calculus we have developed in a series of papers [5–8]. This work is perhaps best characterized as being an abstract epistemology which, though based upon conventional mathematics of real analysis and metric geometry, by necessity has novel conventions and nomenclature. We shall review as much of this work as is required to make the present paper self-contained.

2. DESCRIPTION SPACE

We start then not with physics but with an abstract mathematical structure which we have named description space (D-space). This has been treated in detail in [5] and [6], and the salient features will be presented here without proofs.

Let V be the vector space $\mathbb{R}^n$ and V^* the dual space. For $\mathbf{F} \in V$ and $\mathbf{a} \in V^*$, the dyad $\mathsf{R} = \mathbf{aF}$ is a bilinear mapping from $V^* \times V$ to $\mathbb{R}$ defined by a double-dot product† $\mathsf{R}(\mathbf{a}', \mathbf{F}') = (\mathbf{aF}):(\mathbf{a}'\mathbf{F}') = (\mathbf{a}\cdot\mathbf{a}')(\mathbf{F}\cdot\mathbf{F}')$. The set of all dyadics ($\mathsf{R} = \mathbf{a}^i\mathbf{F}_i$: Einstein summation over finite index sets, $i = 1, 2, \ldots, m$, $\mathbf{a}^i \in V^*$, $\mathbf{F}_i \in V$) is identical to the vector space of (1, 1) tensors, $T_1^1(V) = V \otimes V^*$. Moreover, $T_1^1(V)$ is a metric space with an inner product of $\mathsf{R} = \mathbf{a}^i\mathbf{F}_i$ and $\mathsf{S} = \mathbf{b}^j\mathbf{G}_j$ given by

$$\begin{aligned} \mathsf{R}:\mathsf{S} &= (\mathbf{a}^i\mathbf{F}_i):(\mathbf{b}^j\mathbf{G}_j) \\ &= (\mathbf{a}^i\cdot\mathbf{b}^j)(\mathbf{F}_i\cdot\mathbf{G}_j) \end{aligned} \tag{1}$$

and with norm

$$\begin{aligned} |\,\mathsf{R}\,|^2 &= \mathsf{R}:\mathsf{R} \\ &= L^{ij}(\mathbf{F}_i\cdot\mathbf{F}_j) \geq 0, \end{aligned} \tag{2}$$

where $L^{ij} \equiv (\mathbf{a}^i\cdot\mathbf{a}^j)$.

Thus we discover that the norm on $T_1^1(V)$ looks very much like the norm on a radius vector, $\mathbf{R} = \mathbf{e}^i x_i$, in skew, rectilinear coordinates: i.e. $|\,\mathbf{R}\,|^2 = g^{ij}x_ix_j$, with $g^{ij} = (\mathbf{e}^i\cdot\mathbf{e}^j)$. The analogy between R and $\mathbf{R}$ is interesting and will be developed further; it is central to this paper. However, a great difference between them must be pointed out. The "components" $\mathbf{F}_i$, analogous to the x_i, are vectors, not scalars. The "coordinates" $\mathbf{a}^i$, analogous to the coordinates $\mathbf{e}^i$, are not a basis for $T_1^1(V)$, which has dimension (dim $V)^2$. Not only that, in saying that the set of *all* dyadics $\mathsf{R} = \mathbf{a}^i\mathbf{F}_i$ is identical to $T_1^1(V)$, we did not specify any definite set $\{\mathbf{a}^i\}$ as coordinates, as one does in choosing a basis set $\{\mathbf{e}^i\}$. What happens if we do choose a definite set $\{\mathbf{a}^i\}$?

† In [6] the scope is enlarged to include dyads over a Hilbert space H. For $x, y \in H$ and $a, b \in H^*$, $\mathsf{R} = by$ is the bilinear mapping from $H^* \times H$ defined by $\mathsf{R}(a, x) = \langle b, a\rangle^*\langle y, x\rangle$.

For a given set of vectors $\{\mathbf{a}^i\}$, the dyadics $\mathbf{R} = \mathbf{a}^i\mathbf{F}_i$, for $\mathbf{F}_i$ ranging freely in V, form a linear subspace of $T^1_1(V)$, and therefore form a metric space with inner product (1) and norm (2). The metrical structure of this space is identical to the linear mathematical formalism (i.e. phenomenology) known as irreversible thermodynamics. This was demonstrated in detail in [5], and we shall give here only a summary sufficient for present purposes. There we presented a general phenomenology and then showed that irreversible thermodynamics was a representative, and by no means unique, example. We motivate this phenomenology using the concept of response as a two-fold condition. A response is characterized by the specification of the causal action and of those constitutive properties of the system which are agents of mediation between action and response. This can be put into precise mathematical terms using dyadics, and we reproduce below the three postulates[5] which underlie what we dubbed "the phenomenological calculus."

Postulate 1. A given system is characterized by a set of vectors $\mathbf{a}^i$ ($i = 1, 2, \ldots, m$) in (the dual space of) $\mathbb{R}^n$, and this set depends upon the physical constitution of the system. As far as describing the dynamic response of the system to the imposition of a set of forces (or more generally, *causes*) $\mathbf{F}^i$ in $\mathbb{R}^n$, they form a complete set of constitutive parameters. The index i denotes subsystems (monads, elements, molecular species, organ systems, etc.) of the system.

Postulate 2. The system dynamics are characterized phenomenologically by the dyadic response tensor $\mathbf{R} = \mathbf{a}^i\mathbf{F}_i$.

Postulate 3. The space spanned by $\mathbf{R}$—i.e. for fixed covariant vectors $\mathbf{a}^1, \ldots, \mathbf{a}^m$ in $V^* \cong \mathbb{R}^n$, the set $D = \{\mathbf{R} = \mathbf{a}^i\mathbf{F}_i : \mathbf{F}_i \in V = \mathbb{R}^n\}$—is called the description space, and $\mathbf{R}$ is invariant under coordinate transformations in D.

3. THE PHENOMENOLOGICAL CALCULUS

These three postulates do not establish a direct phenomenological connection between causes and effects. This connection is provided by the geometrical structure of description spaces. We *define* effects to be the duals (in D-space) of the causes, $\mathbf{F}_i$. The set of effects are denoted by $\{\mathbf{F}^j\}$, and to make a sharper distinction (and to prepare for the physical interpretation) we shall use a different letter for effects: $\mathbf{J}^j \equiv \mathbf{F}^j$. To find $\mathbf{J}^j$ in the dual D-space, we use the fact that $\mathbf{R} = \mathbf{a}^i\mathbf{F}_i$ is a linear mapping $V^* \to V^*$, defined by

$$\begin{aligned} \mathbf{R}(\mathbf{a}^j,\cdot) &= (\mathbf{a}^i\mathbf{F}_i)(\mathbf{a}^j\cdot) \\ &= (\mathbf{a}^i\cdot\mathbf{a}^j)\mathbf{F}_i \\ &= L^{ij}\mathbf{F}_i \\ &\equiv \mathbf{J}^j(= \mathbf{F}^j). \end{aligned} \tag{3}$$

We see that the L^{ij} of (2) are like a metric tensor, raising indices of components in D-space. To see how coordinates transform, we use the invariance of $\mathbf{R}$; from

$$\mathbf{R} = \mathbf{a}^i\mathbf{F}_i = \mathbf{a}_j\mathbf{J}^j \tag{4}$$

we have

$$\mathbf{a}^i\mathbf{F}_i - \mathbf{a}_jL^{ji}\mathbf{F}_i = 0 \tag{5}$$

or

$$\mathbf{a}^i = L^{ij}\mathbf{a}_j \tag{6}$$

since $L^{ij} = L^{ji}$ by the symmetry of the dot-product. The metric geometry of this simple phenomenological calculus can be succinctly expressed in the following duality-invariance-diagram (DID):

$$\begin{array}{ccc} \mathbf{F}_i & \xrightarrow{\quad L^{ij} \quad} & \mathbf{J}^j \\ & \mathbf{a}^i\mathbf{F}_i = \mathbf{R} = \mathbf{a}_j\mathbf{J}^j & \\ \mathbf{a}^i & \xleftarrow[\quad L^{ij} \quad]{} & \mathbf{a}_j \end{array} \quad . \tag{7}$$

In previous papers we have presented numerous examples of physical realizations of the phenomenology expressed in the DID (7), including enzyme-substrate recognition[6], R. Rosen's metabolic-repair (M-R) systems[8], stoichiometry[7] and hierarchical systems in general[9]. G. Rosen[10] has extended these ideas to the optimization of many-parameter systems. Its realization in irreversible thermodynamics is immediate. The central premise of irreversible thermodynamics[3] is that if one can by means of a physical model calculate the dissipation function so that it is a bilinear function of forces and fluxes of molecular species i (i.e. $\delta = \mathbf{F}_i \cdot \mathbf{J}^i$), then these forces and fluxes are linearly related (i.e. $\mathbf{J}^i = L^{ij}\mathbf{F}_j$). The Onsager symmetry mentioned in Section 1 is given by the reciprocity equations $L^{ij} = L^{ji}$. The correspondence to the DID (7) is obvious. It must be noted that though there is a correspondence in form, there is a significant difference in physical approach. According to the particular transport system to be analysed, we choose a set of forces, $\mathbf{F}_i$. Conjugate to these forces is a set of constitutive parameters, $\mathbf{a}^i$, which characterize the response of the system to the given forces. The fluxes, $\mathbf{J}^j$, are by definition the components dual to the $\mathbf{F}_i$: thus $\mathbf{J}^j \equiv \mathbf{F}^j = L^{ij}\mathbf{F}_i$ by (3). The squared norm of the response tensor is given by (3) and (2) as $| \mathbf{R} |^2 = \mathbf{F}_i \cdot \mathbf{J}^i \geq 0$. But this is the equation for the dissipation function, and so we make the correspondence $\delta \equiv | \mathbf{R} |^2$. The premise that all products $\mathbf{F}_i \cdot \mathbf{J}^i$ (no summation) must carry the same physical units (i.e. of dissipation) is seen from our viewpoint to be necessary to assure that D-space is a metric, not an affine space. The Onsager conditions are merely a mathematical consequence of the symmetry defined into the metric $L^{ij} \equiv \mathbf{a}^i \cdot \mathbf{a}^j$.

Thus the metric geometry of description space is identical to linear irreversible thermodynamics, with the rather surprising result that the Second Law as expressed by $\delta \geq 0$ corresponds to the positive-definite condition of the norm $| \mathbf{R} |^2 \geq 0$. This must not be misconstrued as a *proof* of the Second Law. All that is claimed is that irreversible thermodynamics is homomorphic to a certain metric space. In Section 9 we shall likewise demonstrate that Dirac's quantum mechanics is homomorphic to D-space.

4. AGING AND INTRINSIC TIME

The phenomenological calculus shown schematically in the DID (7) actually has a great deal more structure than classical irreversible thermodynamics. Its metric properties suggest two natural extensions of the classical theory: (a) nonlinear coupling and (b) the generation of a time intrinsic to the dynamics. Just as the transition from Euclidean to Riemannian geometry is accomplished by allowing the metric tensor to depend upon the components [i.e. $g^{ij}(x) = \mathbf{e}^i(x)\cdot\mathbf{e}^j(x)$], a nonlinear irreversible thermodynamics can be created by allowing the metric tensor in D-space to depend upon the components [i.e. $L^{ij}(\mathbf{F}) = \mathbf{a}^i(\mathbf{F})\cdot\mathbf{a}^j(\mathbf{F})$]. In effect, this makes a Riemannian description space. We shall not explore the ramifications of allowing $\mathbf{a}^i = \mathbf{a}^i(\mathbf{F})$ but will instead turn our attention to the nonlinear D-space created by allowing the coordinates to depend upon a time intrinsic to irreversible systems which can be represented by the DID (7).

The discussion will be facilitated by defining *aging*. Aging is an irreversible change in the physical constitution of a real system brought about by dynamical processes. Actually, this is not a practical definition in so far as it does not define a unique observable called age; nevertheless, it does serve a useful purpose. It helps focus our intuition. It excludes from consideration the commonly used expression "aged system," which refers to an asymptotic approach to equilibrium in damped systems (where, with few exceptions, one assumes that no irreversible changes in the constitutive parameters occur; attention is upon the exhaustion of available energy). A moment's reflection shows that it is quite difficult to frame an operational definition of age, especially in complex systems with many interacting subsystems. Biologists, for example, have had as much difficulty with this as with their attempts to define *life* or *fitness* in operational terms. We shall not reject the concept of age, but will restrict our analysis to the concept of intrinsic time, which we can define in precise mathematical terms within the context of the phenomenological calculus.

The mathematically imprecise, but intuitively appealing phrase "entropy determines the arrow of time," coupled with the concept in relativity of a *proper time* defined on the metrical structure of space-time, leads us to define an *intrinsic time*, as

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 \equiv |\mathbf{R}|^2 = L^{ij}\mathbf{F}_i\cdot\mathbf{F}_j = \mathbf{F}_i\cdot\mathbf{J}^i = \delta \geq 0. \tag{8}$$

Thus intrinsic time is generated by the various dissipative processes of the system, each subsystem (indexed by i) contributing in proportion as $\mathbf{F}_i\cdot\mathbf{J}^i$ (no summation). These partial contributions to dissipation, and hence to the generation of τ, are elements in a metric space, and their sum, $\mathbf{F}_i\cdot\mathbf{J}^i = L^{ij}\mathbf{F}_i\cdot\mathbf{F}_j$, endows intrinsic time with the metrical structure of the DID (7). The "metric" of time is thus the same metric that provides the couplings between forces and fluxes. The units of intrinsic time are not simple, since the dissipation function, δ, is the entropy production per unit time multiplied by temperature and has the units of power.

Intrinsic time is well defined by (8) in terms of quantities accessible to measurement but is not actually what we observe in an aging system. What we observe is a progressive change in the constitutive properties of the system. A rubber band, for instance, when stretched a great many times loses its elastic properties as internal entropy-producing processes increase cross-linking. It seems therefore natural to propose that the constitutive parameters, $\mathbf{a}^i$, of our phenomenology (and by homomorphism, those of irreversible thermodynamics) have a dynamics in intrinsic time. Since it is a dynamics of aging, it will be nonautonomous, depending explicitly upon τ. Furthermore, just as the dependence of these parameters (coordinates) upon forces (components) leads to nonlinear, Riemannian D-spaces, a dependence of the parameter dynamics upon forces is not *a priori* excluded.

To put this into concrete terms, consider the typical situation in transport theory, with fluxes $\mathbf{J}^i$ and driving forces $\mathbf{F}_i$. There is a dynamics on state variables proceeding in t-time, which we shall call extrinsic time since it is measured by some clock outside and independent of the system. The equation of continuity for the concentration, C_i, of molecular species i gives the *state dynamics:*

$$-\frac{\partial C_i}{\partial t} = \nabla\cdot\mathbf{J}^i = \nabla\cdot L^{ij}\mathbf{F}_j. \tag{9}$$

Expressed here in general terms, there is a corresponding

parameter dynamics:

$$L^{ij} = \mathbf{a}^i\cdot\mathbf{a}^j, \qquad \frac{\mathrm{d}\mathbf{a}^i}{\mathrm{d}\tau} = \mathbf{f}_i(\mathbf{F};\ \tau). \tag{10}$$

The two dynamics above are connected via (8) in a metrical

time dynamics:

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = L^{ij}(\tau)\mathbf{F}_i\cdot\mathbf{F}_j. \tag{11}$$

It is difficult to imagine how one might devise a theory of aging in the context of irreversible thermodynamics without having the additional structure afforded by the phenomenological calculus. We have seen how it provides a natural way to relate an intrinsic time to entropy production. But, even more, no matter how one wished to define τ (including $\tau = \beta t$; β = constant), there is a serious problem which arises when one puts a dynamics at the level of the classical phenomenology based upon the parameters L^{ij}. The Second Law dictates that $\delta = L^{ij}\mathbf{F}_i\cdot\mathbf{F}_j$ be positive definite for any set of forces. Therefore, solutions $L^{ij}(\tau)$ to any aging dynamics are subject to stringent conditions; for example, for two molecular species it is necessary and sufficient that

$$L^{11}(\tau) > 0 \text{ and } L^{11}(\tau)L^{22}(\tau) - [L^{12}(\tau)]^2 > 0 \tag{12}$$

for all values of $\tau(t)$; this also implies $L^{22}(\tau) > 0$. For a system with n species, there are n such conditions, since a necessary and sufficient condition for a bilinear form to be positive definite is that the determinants of all principal minors are positive. In the phenomenological calculus (and hence in an irreversible thermodynamics extended to encompass its metrical structure), which is based upon the parameter set $\mathbf{a}^i$, it is *ipso facto* always true that $\delta \equiv |\mathbf{R}|^2 \geq 0$. Any dynamics on $\mathbf{a}^i(\tau)$ satisfies the Second Law.

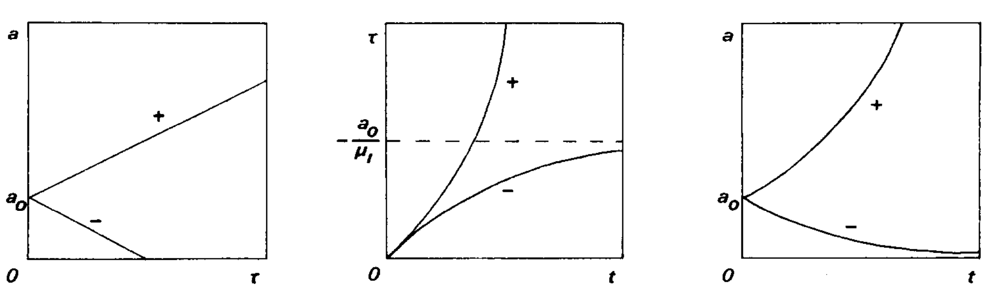

Fig. 1. $+$: $\mu_1 > 0$ and $-$: $\mu_1 < 0$.

5. PARAMETER DYNAMICS AND TIME DYNAMICS

To illustrate the interesting and intriguing behavior of aging systems, we will present several representative examples. For clarity we make the system as simple as possible. It will be in one dimension, have a single species with parameter a, and be acted upon by a single force $F > 0$, assumed to be constant in time. The time dynamics is then just $d\tau/dt = aF$. For the parameter dynamics (10) we shall present various polynomial functions for $f(\tau) = da/dt$.

(a) $f(\tau) = 0$

This parameter dynamics has the solution $a = a_0 =$ constant, and the resulting time behavior $\tau = a_0Ft$. This trivial case conveys an important fact. Intrinsic time (and hence entropy) is not, per se, a measure of aging. In spite of the dissipation, $\delta = a^2F^2$, in this system, nothing has aged. There is no change in the constitution of the system, as characterized by the parameter $a = a_0$. Nothing has "grown old."

(b) $f(\tau) = \mu_1$

The parameter dynamics (10) has the solution $a = a_0 + \mu_1\tau$, which via (11) gives the time equation

$$\tau = \frac{a_0}{\mu_1}(e^{\mu_1 Ft} - 1). \tag{13}$$

The two possible dynamics, for $\mu_1 > 0$ and $\mu_1 < 0$, have strikingly different behaviors, as shown in Fig. 1. For $\mu_1 < 0$, as a goes to zero, the flux, $J = a^2F$, goes to zero, thereby bringing about a cessation in the generation of intrinsic time by dissipation.

(c) $f(\tau) = \mu_2\tau$

In this example, the parameter obeys $a = a_0 + \frac{1}{2}\mu_2\tau^2$, giving the resultant time equation

$$\tau = \sqrt{\frac{2a_0}{\mu_2}}\tan\left(Ft\sqrt{\frac{a_0\mu_2}{2}}\right) \tag{14}$$

for $\mu_2 > 0$. The behavior of this mode of aging is depicted in Fig. 2. The fascinating barrier in extrinsic time disappears when we consider the dynamics for $\mu_2 < 0$: see Fig. 3. In

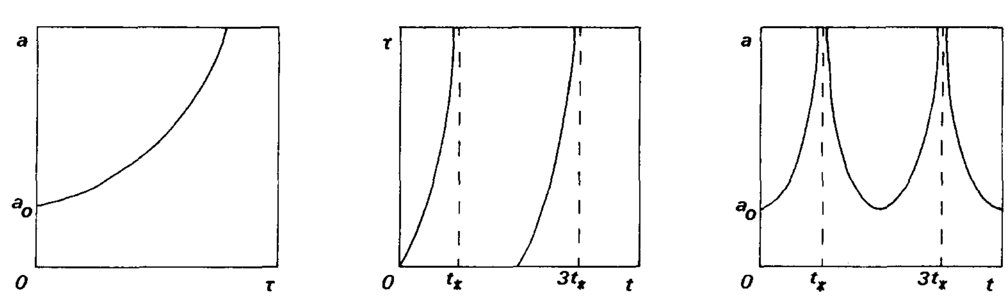

Fig. 2. $\mu_2 > 0$ and $t_* = \pi/F(2a_0\mu_2)^{1/2}$.

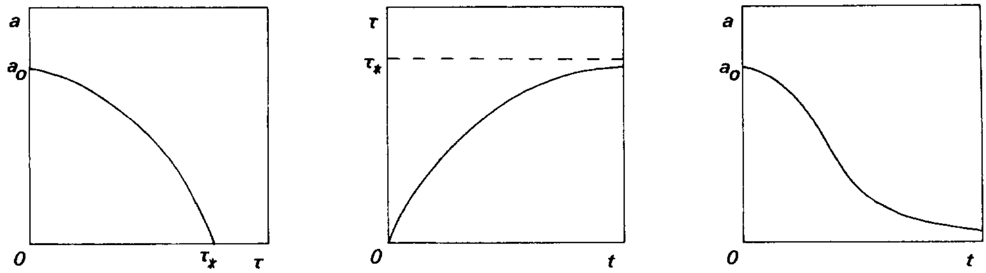

Fig. 3. $\mu_2 < 0$ and $\tau_* = (-2a_0/\mu_2)^{1/2}$.

that case, the time equation is

$$\tau = \sqrt{\frac{-2a_0}{\mu_2}} \tanh\left(Ft\sqrt{\frac{-a_0\mu_2}{2}}\right). \tag{15}$$

One must, of course, apply the usual cautions about considering the implicit and explicit assumptions of hypothetical situations in the interpretation of the asymptotic behavior of the preceding examples. None the less, they make the point that the behavior of aging systems is varied and, at first glance, counter-intuitive.

6. DESCRIPTION SPACES WITH COMPLEX VECTORS

All the examples of the dynamics of parameter aging presented above have solutions $\mathbf{a}^i(\tau)$ or $\mathbf{a}^i(t)$ which are real. There is no *a priori* reason why the dynamics in intrinsic time must be restricted to a limited set of differential equations having as solutions real vectors. In fact, the possibility of having the parameter set $\{\mathbf{a}^i\}$ be complex vectors leads to an interesting and powerful generalization of the phenomenological calculus. In particular, we shall discover a homology in structure between irreversible thermodynamics and quantum mechanics. The step from description spaces with real vectors to those with complex vectors is mathematically simple but immense as regards physical interpretation—taking one from deterministic to stochastic connections between a representation and its dual representation.

In our construction of a DID with complex vectors we shall retain the previous notation. In the dual space, we return to the correspondence $\mathbf{F}^i \equiv \mathbf{J}^i$. Dyads such as $\mathbf{aF}$ will, however, now be denoted by the more explicit tensor notation $\mathbf{a} \otimes \mathbf{F}$. Thus the response tensor is written $\mathbf{R} = \mathbf{a}^i \otimes \mathbf{F}_i = \mathbf{a}_j \otimes \mathbf{F}^j$. Finally, we shall allow vectors in Hilbert space and no longer write vectors in boldface. For precise definitions of the inner products we shall use, the reader should review [6]. The inner product in H^* is denoted $\langle\cdot, \cdot\rangle^*$.

Because the underlying field is now complex and because of the role of the complex conjugate operation (denoted by a bar over a vector or number), inner products are not quite bilinear and symmetric; they are Hermitian forms. That is, they are linear in the first variable and conjugate linear in the second variable. In particular, for any complex number λ,

$$\langle\lambda a^i, b^j\rangle^* = \lambda\langle a^i, b^j\rangle^* = \langle a^i, \bar{\lambda} b^j\rangle^*, \tag{16}$$

$$\langle a_i, \lambda b_j\rangle = \bar{\lambda}\langle a_i, b_j\rangle = \langle\bar{\lambda} a_i, b_j\rangle. \tag{17}$$

However, the response tensor itself remains a bilinear form:

$$(\lambda a^i) \otimes F_i = \lambda(a^i \otimes F_i) = a^i \otimes \lambda F_i$$

and (18)

$$(\lambda a_i) \otimes F^i = \lambda(a_i \otimes F^i) = a_i \otimes \lambda F^i.$$

The complex DID is constructed as follows:

(a)
$$\begin{aligned} F^j &\equiv \mathbf{R}(a^j, \cdot) = (a^i \otimes F_i)a^j = \langle a^i, a^j\rangle^* F_i \\ &= L^{ij}F_i \quad \text{with} \quad L^{ij} = \langle a^i, a^j\rangle^* = \overline{L}^{ji}, \end{aligned} \tag{19}$$

(b)
$$\begin{aligned} |\,\mathbf{R}\,|^2 &= (a^i \otimes F_i):(a^j \otimes F_j) = \langle a^i, a^j\rangle^*\langle F_i, F_j\rangle \\ &= L^{ij}\langle F_i, F_j\rangle \\ &= \langle L^{ij}F_i, F_j\rangle = \langle F^j, F_j\rangle \\ &= \langle F_i, \overline{L}^{ij}F_j\rangle = \langle F_i, L^{ji}F_j\rangle = \langle F_i, F^i\rangle, \end{aligned} \tag{20}$$

(c)
$$a^i \otimes F_i = a_j \otimes F_j = a_j \otimes L^{ij}F_i = L^{ij}a_j \otimes F_i$$

so

$$(a^i - L^{ij}a_j) \otimes F_i = 0$$

giving

$$a^i = L^{ij}a_j = \overline{L}^{ji}a_j. \tag{21}$$

Thus we see that the complex DID has the same morphology as the real DID, except that the Onsager reciprocal relations read $L^{ij} = \overline{L}^{ji}$. Note also the position of the summed index in (19) and (21).

Looking at the squared norm, we see that

$$\begin{aligned} |\,\mathbf{R}\,|^2 &= \langle a^i, a^j\rangle^*\langle F_i, F_j\rangle \\ &= \sum_i |\,a^i\,|^2\,|\,F_i\,|^2 + 2\sum_i\sum_{j>i} \mathrm{Re}\,\langle a^i, a^j\rangle^*\langle F_i, F_j\rangle \end{aligned} \tag{22}$$

since $\langle a^i, a^j\rangle^*\langle F_i, F_j\rangle = \overline{\langle a^j, a^i\rangle^*\langle F_j, F_i\rangle}$. Therefore, $|\,\mathbf{R}\,|^2$ is real. Not only that, but $|\,\mathbf{R}\,|^2 \geq 0$, and the complex description space is indeed an inner product space. To establish this requires only slight modification to the proof for real D-spaces given in [5].

7. THE WAVE DID

There is no end to the dynamics which result in a complex-valued set of parameters $\{a^i\}$. The task at hand is to find examples with interesting and useful interpretations, suitable for the representation of natural systems. As our first example, we shall consider a simple wave of frequency ω, amplitude A, and phase ϕ. We shall start with its DID and then proceed to the dynamics which generates the parameters (i.e. coordinates) of this wave DID. The vector representation $\mathbf{R} = A\,\mathrm{e}^{\mathrm{i}(\omega t+\phi)}$ is transformed to a response-tensor representation by the correspondence

$$\mathbf{R} = A\, e^{i(\omega t + \phi)} \Leftrightarrow \mathbf{R} = e^{i\omega t} \otimes A\, e^{i\phi}. \tag{23}$$

For the superposition of several waves, we have $a^i = e^{i\omega_i t}$ and $F_i = A_i\, e^{i\phi_i}$. Note that with the multiplicity of subscripts, the Einstein summation convention can no longer be employed. The response tensor is therefore written as

$$\mathbf{R} = \sum e^{i\omega_i t} \otimes A_i\, e^{i\phi_i}. \tag{24}$$

The classic example of wave interference is Young's double-slit experiment. The two emergent waves have the same amplitude, A, the same frequency, ω, and phase difference $(2\pi d/\lambda) \sin \Theta$, where d is the distance between the slits, Θ is the azimuth direction to the display screen, and λ is the wave length. By (24) and (22), the interference pattern has an amplitude pattern

$$|\,\mathbf{R}\,| = 2A \cos\left(\frac{\pi d}{\lambda} \sin \Theta\right). \tag{25}$$

This is the first intimation that complex D-spaces may provide stochastic interpretations of phenomena. We shall return to this idea after considering the intrinsic time associated with a wave DID.

8. INTRINSIC TIME FOR THE WAVE DID

We shall work backwards to find the intrinsic time scale associated with the dynamics giving rise to the parameter set $a^i = e^{i\omega_i t}$. The dynamics in extrinsic time is obviously

$$\frac{da^i}{dt} = i\omega_i a^i \quad \text{(no summations here)} \tag{26}$$

with the normalization $a_i(0) = 1$. The second-order differential equation for a harmonic oscillator, with its two constants of integration, is not required for the dynamics because amplitude and phase information for the wave response tensor are carried in the components, F_i.

In this system there is no dissipation, and therefore there is no single, entropy-related variable with which to reparameterize t-time. Thermodynamical intuition cannot be used here, as it was before, as the guide for choosing an intrinsic time. We must turn to the well-known concepts of dimensional analysis. In fact, we have already considered[11] the problem of defining an intrinsic time scale for dynamics similar to (26). One simply casts (26) into dimensionless form,

$$-ia^{-1}\frac{da}{d\tau} = 1, \tag{27}$$

to find the transformation $\tau_i = \omega_i t$. This seems to be a reasonable transformation. Two oscillators having generated the same amount of intrinsic time, $\tau_i = \tau_j$, will have passed through periods of t-time standing in the relation $\omega_i t_i = \omega_j t_j$.

As mentioned above, there is no single, intrinsic time, τ, for the composite system. Each oscillator (i.e. wave) has its own time, τ_i. Here we lose the metrical structure of intrinsic time provided by the metrical structure of the DID representing irreversible thermodynamics. We still have a norm $|\mathbf{R}|^2$, but as there is no dissipation we have no physical motivation for relating it to the generation of intrinsic time as in (8) and (11). Is it possible to find a similar physical interpretation for the time transformation of (26) to (27)?

Being nondissipative does confer upon a system a singular virtue: that of having a conserved energy, which in the case of a harmonic oscillator goes as ω^2. Thus, the intrinsic time of (27) scales the sub-systems proportionate to their energy. Indeed, it is a commonplace observation that energy determines the characteristic time scales of physical systems. There is a hierarchy of characteristic time scales, going from the very short scale of nuclear processes mediated by the strong nuclear force to the cosmic time scale of the evolution of structures under the influence of the weak gravitational force.

The role of energy in the transformation from extrinsic time to intrinsic time becomes explicit if one considers waves with an immediate physical meaning: namely, de Broglie matter-waves. For a particle moving along the x-axis with momentum p and energy E, the response tensor for its wave representation is

$$\begin{tikzcd} F = A\, e^{ipx/\hbar} \arrow[dr] & \\ & \mathbf{R} = e^{iEt/\hbar} \otimes e^{ipx/\hbar} \\ a = e^{iEt/\hbar} \arrow[ur] & \end{tikzcd} \tag{28}$$

using the Planck–Einstein relations, $E = h\nu$ and $p = h/\lambda$ for the frequency ν and the wavelength λ.

The decomposition of the de Broglie matter-wave (28) into a coordinate, a, and a component, F, of a response tensor corresponds to the division in the measuring process most often proposed to explain the two Heisenberg uncertainty relations. A measurement is accomplished by the superposition of (28) with a standard wave, the minimal (and hence most precise) reading being one beat. Thus the energy difference ΔE and the time interval must satisfy $\Delta E \Delta t \geq h$, which gives an uncertainty at the level of coordinates in the D-space representation. Likewise, the single-beat criterion applied to a momentum difference Δp and a space interval gives an uncertainty of $\Delta p \Delta x \geq h$ at the level of components. Rather than continuing further with matter-waves, we shall take the logical next step and proceed on to quantum mechanics.

9. THE QUANTUM MECHANICS DID

To find the quantum mechanics response tensor, it is easiest to present the usual Dirac representation and then point out the correspondences: see [12], Chap. 3, for a lucid summary of the basic postulates of quantum mechanics. Every measurable physical quantity is described by an operator A. The results of measurement are the eigenvalues, a_n, of A. The operator A is assumed to be Hermitian, and so the a_n are real. For a discrete, nondegenerate spectrum, the eigenvectors of A form a basis, and any state vector $|\psi\rangle$ can be represented by

$$|\psi\rangle = \sum_n c_n \, |u_n\rangle, \tag{29}$$

where

$$A\,|u_n\rangle = a_n\,|u_n\rangle \tag{30}$$

and

$$c_n = \langle u_n \mid \psi \rangle. \tag{31}$$

The probability $P(a_n)$ that a_n is the result of a measurement is

$$P(a_n) = | c_n |^2, \tag{32}$$

where we have assumed a suitable normalization of states. Since A is Hermitian, there is a dual space of eigenvectors $\langle u_n \mid$ having the same eigenvalues.

In the context of a complex description space, the ket and bra vectors, $\mid u_n \rangle$ and $\langle u_n \mid$, correspond to coordinates a^i and a_i. To make the correspondence clear, we relabel our coordinates: $a^i \equiv u^i$ and $a_i \equiv u_i$. The coefficients, c_n, correspond to components, which we denote $F_i \equiv c_i$ and $F^i \equiv c^i$. Instead of a state vector, $\mid \psi \rangle$, we form a state tensor, $\boldsymbol{\Psi}$, given as the following response tensor:

$$\boldsymbol{\Psi} = a^i \otimes F_i = u^i \otimes c_i = a_i \otimes F^i = u_i \otimes c^i \tag{33}$$

with the DID

$$\begin{tikzcd} c_i \arrow[rr, "L^{ij}"] \arrow[dr, dash] & & c^i \arrow[dl, dash] \\ & \boldsymbol{\Psi} \arrow[dl, dash] \arrow[dr, dash] & \\ u^i & & u_i \arrow[ll, "L^{ij}"'] \end{tikzcd} \tag{34}$$

Recall that in (19) and (21) we saw that the D-space metric tensor, L^{ij}, raises indices in components and in coordinates by summing over the first index and the second index, respectively. The algebra of Dirac's ket and bra vectors is just that of (16)–(18) except that, in general, F_i and F^i can be vectors whereas in the quantum mechanics DID (34) they are complex numbers.

To find the metric tensor, L^{ij}, (or coupling coefficients as is said in irreversible thermodynamics) we use $\boldsymbol{\Psi}$ as the mapping $\boldsymbol{\Psi}$: $V \to V$. Thus

$$\begin{aligned} \boldsymbol{\Psi}(u^j, \cdot) &= (u^i \otimes c_i)(u^j, \cdot) = \langle u^i, u^j \rangle^* c_i \\ &= L^{ij} c_i \\ &\equiv c^j. \end{aligned} \tag{35}$$

However, the eigenvectors u^i are orthonormal; so

$$L^{ij} \equiv \langle u^i, u^j \rangle^* = \delta^{ij} \tag{36}$$

and

$$c^i = \delta^{ij} c_j. \tag{37}$$

Finally, we must consider the norm of $\boldsymbol{\Psi}$; it is given by the double inner product as

$$\begin{aligned} |\,\boldsymbol{\Psi}\,|^2 = \boldsymbol{\Psi}:\boldsymbol{\Psi} &= L^{ij}\langle c_i, c_j\rangle \\ &= \sum_i |\,c_i\,|^2 \\ &= \sum_i P(a_i) \end{aligned} \tag{38}$$

by the definition (32) of the probability $P(a_i)$.

Therefore, we conclude that quantum mechanics is structurally homomorphic to irreversible thermodynamics, where in the first instance $|\,\mathbf{R}\,|^2 = |\,\boldsymbol{\Psi}\,|^2$ gives the probabilities of events and in the second instance $|\,\mathbf{R}\,|^2 = \delta$ gives the dissipation. It is interesting that irreversible thermodynamics has exploited more of the structure available in that its components, $\mathbf{F}_i$, are vectors, not merely numbers, and that it does not have a theory of measurement that leads to decoupling, that is, to $L^{ij} = \delta^{ij}$. Furthermore, in irreversible thermodynamics the coordinates, $\mathbf{a}^i$, do not necessarily form a set of independent vectors. The price paid is that L^{ij} does not necessarily possess a unique inverse, which implies that one cannot go from effect to cause[5].

The reader is referred to [7] for an earlier discussion on the connections between irreversible thermodynamics, quantum mechanics and the phenomenological calculus. Particular emphasis was placed upon the spectral resolution of self-adjoint operators using projection operators in D-space.

10. INTERFERENCE PHENOMENA

Using the formalism developed above, the quantum mechanical analysis of interference is straightforward. In Young's experiment, for example, the preparation of the system by the interposition of two slits between source and screen puts the system into a state which is a linear superposition of the two possible paths. Thus, in Dirac's notation

$$\begin{aligned} |\,\psi\rangle &= \lambda_1\,|\,\psi_1\rangle + \lambda_2\,|\,\psi_2\rangle, \\ |\,\lambda_1\,|^2 &+ |\,\lambda_2\,|^2 = 1, \end{aligned} \tag{39}$$

or in ours,

$$\boldsymbol{\Psi} = \psi^1 \otimes \lambda_1 + \psi^2 \otimes \lambda_2. \tag{40}$$

The measurement of the interference pattern at the screen is associated with an observable A, with eigenvalues a_i, and eigenvectors u^i. The DID of this phenomenon is given below:

$$\begin{tikzcd} \lambda_i \arrow[rr, "M^{ij}\ (i = 1, 2)"] \arrow[dr, dash] & & c^j \\ & \psi^1 \otimes \lambda_1 + \psi^2 \otimes \lambda_2 = \boldsymbol{\Psi} = u_j \otimes c^j \arrow[ur, dash] \arrow[dr, dash] & \\ \psi^i \arrow[ur, dash] & & u_j \arrow[ll, "M^{ij}\ (i = 1, 2)"'] \end{tikzcd} \tag{41}$$

Recalling that $\boldsymbol{\Psi}$ is a mapping, we obtain

$$\begin{aligned} \boldsymbol{\Psi}(u^j, \cdot) &= (\psi^1 \otimes \lambda_1 + \psi^2 \otimes \lambda_2)(u^j, \cdot) \\ &= \langle \psi^1, u^j \rangle^* \lambda_1 + \langle \psi^2, u^j \rangle^* \lambda_2 \\ &= M^{1j}\lambda_1 + M^{2j}\lambda_2 \\ &\equiv c^j, \end{aligned} \tag{42}$$

where $M^{ij} = \langle \psi^i, u^j \rangle^*$ is a coupling coefficient of the "mixed" representation. Likewise, we have

$$\begin{aligned} \boldsymbol{\Psi}(u_j, \cdot) &= M_{1j}\lambda^1 + M_{2j}\lambda^2 \\ &\equiv c_j. \end{aligned} \tag{43}$$

Note that M^{ij} and M_{ij} are not conjugate-symmetric, since i and j have different ranges. Also recall the summation conventions of (19) and (21) as regards components and coordinates.

The various probabilities of obtaining a_i in a measurement are, by (38), contained in the squared norm $|\,\boldsymbol{\Psi}\,|^2$, which is

$$\begin{aligned} |\,\boldsymbol{\Psi}\,|^2 &= \langle c_i, c^i \rangle \\ &= \sum_i \{|\,\lambda_1\,|^2 \langle M^{1i}, M_{1i} \rangle + |\,\lambda_2\,|^2 \langle M^{2i}, M_{2i} \rangle + \langle \lambda_1, \lambda^2 \rangle \langle M^{1i}, M_{2i} \rangle \\ &\qquad + \langle \lambda_2, \lambda^1 \rangle \langle M^{2i}, M_{1i} \rangle\} \\ &= \sum_i \{|\,\lambda_1\,|^2\,|\,M^{1i}\,|^2 + |\,\lambda_2\,|^2\,|\,M^{2i}\,|^2 + 2\,\mathrm{Re}\,\lambda_1\bar{\lambda}_2 \langle M^{1i}, M_{2i} \rangle\} \\ &= \sum_i P(a_i). \end{aligned} \tag{44}$$

11. INTRINSIC TIME AND THE SCHRÖDINGER EQUATION

The quantum mechanics basis vectors, u^i, are analogous to the constitutive parameters, $\mathbf{a}^i$, of irreversible thermodynamics, both being coordinates in their respective description spaces. In irreversible thermodynamics, the constitutive parameters have an aging dynamics (10) in an intrinsic time (8) defined by the dissipation function, $|\,\mathbf{R}\,|^2 \equiv \delta$. The representations of quantum systems are in complex D-spaces, and now we shall show that the quantum mechanics basis vectors have a dynamics similar to the wave dynamics (27), with an intrinsic time scaled by energy.

Any discussion of time in the context of quantum mechanics immediately calls to mind the Schrödinger equation, which governs the time evolution of the state vector, $|\,\psi(t)\rangle$. The Hamiltonian, H, is the observable associated with the total energy of the system. To show clearly a fundamental relationship between energy and extrinsic time, the Schrödinger equation can be presented as

$$H\psi = E\psi = \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t}\psi, \tag{45}$$

where energy, E, is identified with the operator $E \rightarrow \mathbf{i}\hbar\, \mathrm{d}/\mathrm{d}t$. As will be seen, the Hamiltonian is associated with intrinsic time. Multiplying (operating on the left) by $\psi^* = \bar{\psi}$, one obtains

$$\psi^* H \psi = \mathbf{i}\hbar\psi^* \frac{\mathrm{d}\psi}{\mathrm{d}t}. \tag{46}$$

Since the left-hand side is a positive definite form such that $\psi^* H \psi \geq 0$, the right-hand side is also

$$\mathbf{i}\hbar\psi^* \frac{\mathrm{d}\psi}{\mathrm{d}t} \geq 0. \tag{47}$$

We want to demonstrate that (46) is an aging equation and therefore wish to establish an analogy with dissipative systems. To make it easier to see the analogy, we borrow notation for $| \mathbf{R} |^2$ and set $\psi^*\psi \equiv \delta$. In Section 9 we saw that the quadratic form $\psi^*\psi$ has, in D-space, the metric tensor $L^{ij} = \langle \psi^i, \psi^j \rangle^* = \delta^{ij}$. By the quadratic form $\psi^* H \psi \equiv \delta_H$, the Hamiltonian induces a modified metric $L_H^{ij} = \langle \psi^i, H\psi^j \rangle^*$. Of course, an analogy must consist of more than just borrowing notation. We shall use (47) and (46) to relate δ_H to an analogue of entropy production.

In classical thermodynamics the microscopic definition of entropy is

$$S = k \ln W, \tag{48}$$

where k is Boltzmann's constant, and W is the probability of the state in question. Setting $W = | \psi |^2$, we have an entropy function

$$S = k \ln | \psi |^2 = 2k \ln | \psi |. \tag{49}$$

The associated "dissipation" function is

$$T \frac{\mathrm{d}S}{\mathrm{d}t} = 2kT | \psi |^{-1} \frac{\mathrm{d} | \psi |}{\mathrm{d}t}, \tag{50}$$

where T is temperature.

To go from thermodynamics to quantum mechanics in the preceding expression, we make a correspondence for scaling

$$2kT \rightarrow \mathbf{i}\hbar \tag{51}$$

and another for "unitariness" (inverse $\rightarrow$ adjoint)

$$| \psi |^{-1} \rightarrow \psi^*. \tag{52}$$

We can therefore conclude that these correspondences used in (50) show that indeed (47) is a quantum analogue for dissipation; our convention $\psi^* H \psi = \delta_H$ is thus justified. In the context of the metrical structure of two homologous D-spaces, it is of little significance that δ in irreversible thermodynamics has units of power whereas δ_H in quantum mechanics has units of energy. The similarities are far greater than the differences. This is particularly evident in their role in the generation of intrinsic time.

By analogy to (8) the metric equation for intrinsic time in quantum systems is

$$\left(\frac{d\tau}{dt}\right)^2 = \delta_H = \psi^* H \psi. \tag{53}$$

Schrödinger's equation (46) in τ-time is hence

$$i\hbar\psi^* \frac{d\psi}{d\tau} = \delta_H^{1/2}, \tag{54}$$

which resembles the parameter dynamics (27) for simple waves. The explicit dependence of (54) upon δ_H allows one to regard the Schrödinger equation as an aging dynamics in intrinsic time, with ψ being a "constitutive" parameter (coordinate) in a D-space representation.

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