A Phenomenological Calculus of *Wiener* Description Space

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We dedicate this paper to the memory of our friend *Robert Rosen* (1934–1998)
and to the memory of the Red House (1975–1985)

The phenomenological calculus is a categorical example of *Robert Rosen*'s modeling relation. This paper is an alligation of the phenomenological calculus and generalized harmonic analysis, another categorical example. Our epistemological exploration continues into the realm of *Wiener* description space, in which constitutive parameters are extended from vectors to vector-valued functions of a real variable. Inherent in the phenomenology are fundamental representations of time and nearness to equilibrium.

Prologue. – The phenomenological calculus has its ontologic origins in a *Richardson* and *Rosen* paper [1] on aging and the metrics of time. The three of us, *Robert Rosen*, *I. W. Richardson*, and *A. H. Louie*, once-upon-a-time worked together in the Red House, the quarters of the Biomathematics Program of Dalhousie University in Halifax, Nova Scotia. After the Red House's untimely demise, we jointly published as its memorial the book '*Theoretical Biology and Complexity: Three Essays on the Natural Philosophy of Complex Systems*'. The phenomenological calculus is a direct descendent of the *Richardson* essay [2]. *Rosen*'s Preface in this book describes our personal and scientific relations. In the *Appendix* at the end of the present paper, *I. W. Richardson* gives his personal account on the rise and fall of the Red House, and *Robert Rosen*'s association with it.

In a recently published paper [3] of our phenomenological-calculus sequence, we explained the many connections between this subject matter and *Rosen*'s own work. We shall not repeat the discussion here; the reader is referred to [3] for details.

1. Introduction. – The phenomenological calculus provides mathematical fundamentals of measurement and representation of natural systems. Its versatility has been demonstrated in its connections to a diversity of biological, physical, and chemical topics: the itinerary includes (in the chronological order of the eleven published papers) dissipation [4], aging [5], enzyme–substrate recognition [6], (M,R)-systems [7], chemical dynamics [8], protein modeling [9], quantum mechanics [10], relativity [11], the *Gibbs* paradox [12], membrane transport [13], and anisotropy [3]. For a large class of complex systems, the fundamental duality between cause and effect can be put into precise mathematical terms by a duality between the mathematical representation of the system in the realm of cause and the corresponding representation in the realm of effect. Stated otherwise, the phenomenological calculus is a categorical example of *Robert Rosen*'s modeling relation (see, in particular, *Chapt. 3* of [14]), in every sense of the word *categorical*.

The mathematical object of the phenomenological calculus is the *description space*, D, a subspace of the space $T_1^1(H)$ of type-(1,1) tensors over a real *Hilbert* space, H. Members of D are dyadics called *response tensors*. The connection between cause F_i and effect J^j is mediated by a metric tensor $L^{ij} = \langle a^i, a^j \rangle$, with components defined as the pairwise inner products of the constitutive parameters a^i of the system. We have reviewed the mathematical setting many times before, so we shall not repeat the exercise here. In any case, later on in this paper, we shall discuss the description space over a new *Hilbert* space that we shall construct presently. We now simply mention in passing that the metric geometry of our phenomenological calculus and the invariance of the response tensor may be succinctly expressed in the following arrow diagram, which we call the *duality–invariance diagram* (*Fig. 1*). (The reader is invited to refer to our previously published sequence of papers for details.)

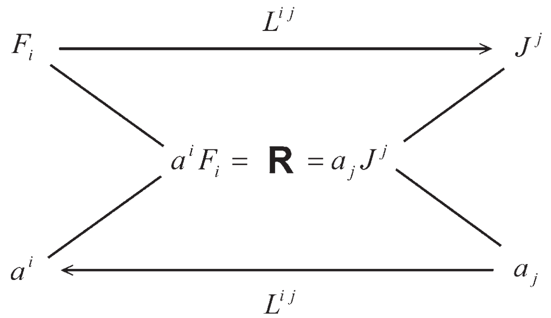

Fig. 1. *Duality–invariance diagram*

While the general *Hilbert*-space formalism has many interesting consequences (especially when $H = L^2$, the space of all square-integrable functions) in our phenomenological calculus, we have demonstrated that a good deal of science is already entailed in the original finite-dimensional Euclidean-space setting when $H = \mathbb{R}^n$. The finite-dimensional setting, however, has a limitation as regards its application to physical systems. This is because part of the phenomenology hinges on, for m constitutive parameters ('species') $\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^m \in \mathbb{R}^n$, the invertibility of the $m \times m$ *Gram* matrix $(L^{ij}) = (\mathbf{a}^i \cdot \mathbf{a}^j)$, which has the same rank k as the dimension of the subspace of $\mathbb{R}^n$ spanned by $\{\mathbf{a}^i\}$. The constraints are $k \leq m$ and $k \leq n$, and (L^{ij}) is invertible if and only if $k = m$. This follows from a linear-algebra theorem (sometimes called '*Gram's* criterion') on *Gram* matrix in inner-product space, and the issue was discussed in detail in [5] and [6].

Putting it alternatively, in $\mathbb{R}^n$, when $m > n$, the set of m constitutive parameters $\{\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^m\}$ cannot be linearly independent, whence the $m \times m$ *Gram* matrix $(L^{ij}) = (\mathbf{a}^i \cdot \mathbf{a}^j)$ is *not* invertible. Thus, in particular, in our physical three-dimensional space $\mathbb{R}^3$, when there are more than three species (*i.e.*, $m > 3$), the *Gram* matrix (L^{ij}) would be singular, whence it would require additional information to go from effect to cause in the phenomenological calculus. In this paper, we generalize the theory to allow for an arbitrary number m of *independent* constitutive parameters, but still retain the physicality of $\mathbf{a}^i \in \mathbb{R}^3$.

2. *Wiener* Space. – Let W be the class of all functions u: $\mathbb{R} \to \mathbb{R}^3$ such that:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|u(t)\|^2 dt < \infty. \tag{1}$$

Since this limit has the expression of *average power*, a function satisfying the finite-limit condition *1* is said to have *finite power*. It is interesting to note that so much, as we shall present below, follows from this single condition *1*. We choose the codomain to be $\mathbb{R}^n$ with $n=3$ because of our dwelling place of three-dimensional physical space. But the theory applies to any finite n, and we often shall, for simplicity, give examples with $n=1$.

The finite-limit condition *1* is a variation on a theme originated from *Norbert Wiener* [15][16] in association with generalized harmonic analysis and the *Fourier* integral. This is why we use the symbol W for this function space, and shall call it *Wiener space*. Note that the 'classical' *Wiener* space, the probability space associated with a *Wiener* measure, is an object altogether different from our usage.

Let L^2 be the *Hilbert* space of all square-integrable functions from $\mathbb{R}$ to $\mathbb{R}^3$, *i.e.*, the class of all functions u: $\mathbb{R} \to \mathbb{R}^3$ such that:

$$\|u\|_2^2 = \int_{-\infty}^{\infty} \|u(t)\|^2 dt < \infty. \tag{2}$$

[(Note the similarities between the finiteness conditions *1* and *2*). Since the integral in *Eqn. 2* is the expression of energy, L^2 functions are said to have *finite energy*.] It is clear that if $u \in L^2$, then:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|u(t)\|^2 dt = 0, \tag{3}$$

whence *a fortiori* $u \in W$. In other words, finite-energy functions have zero power. Thus $L^2 \subset W$.

Let L^2_{LOC} be the *Hilbert* space of all 'locally square-integrable' functions from $\mathbb{R}$ to $\mathbb{R}^3$, *i.e.*, the class of all functions u: $\mathbb{R} \to \mathbb{R}^3$ such that for all $a, b \in \mathbb{R}$:

$$\int_a^b \|u(t)\|^2 dt < \infty. \tag{4}$$

Now $u \notin L^2_{\mathrm{LOC}}$ means there exist $a, b \in \mathbb{R}$ such that $\int_a^b \|u(t)\|^2 dt = \infty$, whence for all $A \leq a$ and for all $B \geq b$, $\int_A^B \|u(t)\|^2 dt = \infty$, implying $\frac{1}{B-A} \int_A^B \|u(t)\|^2 dt = \infty$, so $u \notin W$. Thus, $W \subset L^2_{\mathrm{LOC}}$.

Both inclusions in $L^2 \subset W \subset L^2_{\mathrm{LOC}}$ are proper. For $n=1$, the constant function $u(t) = 1$ is in W, but not in L^2, and the identity function $u(t)=t$ is in L^2_{LOC}, but not in W. The fact that nonzero 'constant functions' are in $W \sim L^2$ is, indeed, why we are investigating W rather than the much more familiar L^2. If $\mathbf{F} \in \mathbb{R}^3$ and $u(t) = \mathbf{F}$ is the constant function, then $u \notin L^2$, but

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|u(t)\|^2 dt = \lim_{T\to\infty} \frac{1}{2T} 2T(\mathbf{F} \cdot \mathbf{F}) = \mathbf{F} \cdot \mathbf{F} < \infty, \tag{5}$$

whence $u \in W$.

3. The *Hilbert* Space *W*. – It is trivial that if $u \in W$ and $\alpha \in \mathbb{R}$, then $\alpha u \in W$. The *Schwarz* inequality and *Lebesgue*'s dominated convergence theorem entail that if u, $v \in W$, then $u + v \in W$. Thus W is a vector space over $\mathbb{R}$.

Define, for $u \in W$:

$$\|u\| = \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|u(t)\|^2 dt \right\}^{\frac{1}{2}}. \tag{6}$$

It will be clear from the context which $\|\cdot\|$ is meant: inside the integral, the $\|\cdot\|$ is that of $\mathbb{R}^3$; *i.e.*, $\|u(t)\|^2 = u(t) \cdot u(t)$ (dot product in $\mathbb{R}^3$). Compare this with the L^2-norm:

$$\|u\|_2 = \left\{ \int_{-\infty}^{\infty} \|u(t)\|^2 dt \right\}^{\frac{1}{2}}. \tag{7}$$

The finiteness condition *1* says that for $u \in W$, $\|u\|^2 < \infty$,whence $\|u\| < \infty$. We saw in *Eqn. 3* that if $u \in L^2 \subset W$, then $\|u\| = 0$. And we also saw in the expression *5* that for a constant function $u(t) = \mathbf{F}$, $\|u\| = \|\mathbf{F}\|$ (where the second $\|\cdot\|$ is that of $\mathbb{R}^3$).

Note that $\|\cdot\|$, as defined in *Eqn. 6*, is only a *pseudo*-norm, because $\|u\| = 0$ for $u \in L^2$. In fact, $\ker\|\cdot\| = L^2$. So, we have to pass onto equivalence classes W/L^2 to obtain a normed linear space. Henceforth, we shall use the same symbol W for the space of equivalence classes of functions. But just as in L^p-spaces, we shall simply speak of the functions themselves rather than of equivalence classes of functions.

Parallel to the proof that L^p-spaces are complete, one may show that W (*i.e.*, W/L^2; see paragraph above) is a complete normed linear space. That is, $(W, \|\cdot\|)$ is a *Banach* space with associated metric:

$$d(u, v) = \|u - v\|. \tag{8}$$

Now define, for u, $v \in W$:

$$\langle u, v \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t) \cdot v(t)\, dt. \tag{9}$$

The *Schwarz* inequality and *Lebesgue*'s dominated convergence theorem again imply that the limit in *Eqn. 9* exists and is finite. It is easy to check that $\langle \cdot, \cdot \rangle$ is an *inner product*, and is related to $\|\cdot\|$ by:

$$\langle u, u \rangle = \|u\|^2 \geq 0. \tag{10}$$

The *Schwarz* inequality in W is:

$$|\langle u, v \rangle| \leq \|u\|\, \|v\|. \tag{11}$$

Thus, *Wiener* space with the inner product of *Eqn. 9*, $(W, \langle \cdot, \cdot \rangle)$, is a *Hilbert* space.

With two constant functions $u(t) = \mathbf{F}$ and $v(t) = \mathbf{G}$, we have:

$$\langle u, v \rangle = \mathbf{F} \cdot \mathbf{G}. \tag{12}$$

This is the generalization we desired. Instead of our physical three-dimensional *Euclidean* space $\mathbb{R}^3$, we now have an infinite-dimensional *Hilbert* space W of functions with *images* in $\mathbb{R}^3$, with a natural extension of the inner product. That members of W are functions with *domain* $\mathbb{R}$ is an added bonus: we now have an intrinsic *time* variable. So, for our new *Wiener* space, causes $F \in W$ and constitutive parameters $a \in W^* = W$ may be dynamic trajectories in addition to static vectors.

Before we continue our phenomenological calculus with type-(1,1) tensors based on this *Hilbert* space W, we shall first consider some functional-analytic aspects of W itself.

4. Autocorrelation. – Let $u \in W$. The *Schwarz* inequality ensures that for every $s \in \mathbb{R}$ the following limit exists:

$$\Gamma_u(s) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t) \cdot u(s+t)\,dt. \tag{13}$$

Here, $\Gamma_u : \mathbb{R} \to \mathbb{R}$ is called the (*time-average*) *autocorrelation function* of u. The average power limit in condition *1* is $\Gamma_u(0)$. This is a trivial connection between correlation and power. We shall discuss a not-so-trivial connection later. Comparing *Eqns. 6* and *13*, we see that:

$$\Gamma_u(0) = \|u\|^2. \tag{14}$$

The following properties (*a–f*) of the autocorrelation function are immediate consequences:

a) For $u \in W$ and $\alpha, s \in \mathbb{R}$:

$$\Gamma_{\alpha u}(s) = \alpha^2 \Gamma_u(s). \tag{15}$$

b) For all $\alpha, s \in \mathbb{R}$:

$$\Gamma_u(s) = \lim_{T\to\infty} \frac{1}{2T} \int_{\alpha-T}^{\alpha+T} u(t) \cdot u(s+t)\,dt. \tag{16}$$

This means that the choice of origin of the domain is unimportant.

c) For $T \in \mathbb{R}$ define:

$$u_T(t) = u(t)\chi_{[-T,T]}(t), \tag{17}$$

where χ_A is the characteristic function of the set A; *i.e.*:

$$\chi_A(t) = \begin{cases} 1, & t \in A \\ 0, & t \notin A \end{cases}, \tag{18}$$

whence:

$$u_T(t) = \begin{cases} u(t), & -T \le t \le T \\ 0, & |t| > T \end{cases}. \tag{19}$$

Then:

$$\Gamma_u(s) = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} u_T(t) \cdot u_T(s+t)\, dt. \tag{20}$$

This says Γ_u may be defined through *truncations* u_T of u.
d) Γ_u is an even function: for all $s \in \mathbb{R}$,

$$\Gamma_u(-s) = \Gamma_u(s). \tag{21}$$

e) For all $s \in \mathbb{R}$,

$$|\Gamma_u(s)| \leq \Gamma_u(0). \tag{22}$$

This is the Γ_u version of the '*Schwarz* inequality'.
f) Γ_u is continuous almost everywhere, and is continuous if and only if it is continuous at 0. Note that Γ_u is not a linear function, but it has a lot of the characteristics of linearity. Property *f* is analogous to that of a linear operator.

The autocorrelation of a constant function $u(t) = \mathbf{F}$ is also constant: $\Gamma_u(s) = \|\mathbf{F}\|^2$.

5. Spectrum. – Let $u \in W$ and Γ_u be its autocorrelation. While their *Fourier* transforms:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(s)\, e^{-ist} ds \tag{23}$$

and:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Gamma_u(s)\, e^{-ist} ds \tag{24}$$

may not exist, the 'integrated *Fourier* transform' of Γ_u:

$$S_u(t) = \lim_{X\to\infty} \int_{-X}^{X} \Gamma_u(s) \left[\frac{e^{-ist} - 1}{-is}\right] ds \tag{25}$$

converges for all $t \in \mathbb{R}$. We call $S_u : \mathbb{R} \to \mathbb{R}$ the *spectrum* of u, and it has the properties *g–l:*
g) $S_u \in L^1_{\mathrm{LOC}}$; *i.e.* for all $a, b \in \mathbb{R}$,

$$\int_a^b |S_u(t)|\, dt < \infty. \tag{26}$$

h) S_u is monotonically increasing.
i) For all $t \in \mathbb{R}$,

$$S_u(t) = \frac{1}{2}[S_u(t+\ ) + S_u(t-)]. \tag{27}$$

This is a nice *Fourier* property. Also, properties *g–i* together say that S_u is a distribution function.
j) $S_u(0) = 0$.
k) $S_u(+\infty)$ and $S_u(-\infty)$ exist, and:

$$S_u(+\infty) - S_u(-\infty) \leq \Gamma_u(0), \tag{28}$$

with equality holding if and only if Γ_u is continuous at 0 (see property f of Γ_u above).
l) The following limit,

$$\lim_{X\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-X}^{X} e^{ist} dS_u(s), \tag{29}$$

converges, in the sense of a '*Cauchy–Riemann–Stieltjes* integral', to a continuous function equal almost everywhere to $\Gamma_u(t)$.

6. Spectral Density. – Let $u \in W$, Γ_u be its autocorrelation, and S_u be its spectrum. Γ_u has *Fourier* transform (in L^1_{LOC}) if and only if S_u is absolutely continuous, whence we may define the *spectral density* of u, $G_u : \mathbb{R} \to \mathbb{R}$, as the derivative of the spectrum:

$$G_u(t) = S'_u(t) \tag{30}$$

at all points where the derivative exists (which is almost everywhere in $\mathbb{R}$). The spectral density has the properties *m–p:*
m) $G_u(t) \geq 0$ almost everywhere.
n) $G_u \in L^1$.
o) Spectrum is the antiderivative of spectral density: for all $t \in \mathbb{R}$,

$$S_u(t) = \int_0^t G_u(s)\, ds. \tag{31}$$

p) The following *Fourier* integral,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G_u(t)\, e^{ist} dt, \tag{32}$$

defines a continuous function equal almost everywhere to $\Gamma_u(s)$. Thus, autocorrelation and spectral density, Γ_u and G_u, form a *Fourier transform pair*. This fact is denoted by the notation:

$$\Gamma_u \leftrightarrow G_u. \tag{33}$$

Explicitly, in terms of *Fourier* integrals,

$$G_u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Gamma_u(s)\, e^{-ist} ds, \tag{34}$$

and:

$$\Gamma_u(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G_u(t)\, e^{ist} dt. \tag{35}$$

The spectral density of a constant function $u(t) = \mathbf{F}$ is an 'impulse', a scalar multiple of the *Dirac* delta function: $G_u(t) = \sqrt{2\pi}\|\mathbf{F}\|^2 \delta(t)$.

7. *Wiener–Khintchine* Theorems. – Let $u \in W$, and for $T \in \mathbb{R}$ let $u_T : \mathbb{R} \rightarrow \mathbb{R}^3$ be a truncation, as defined in *Eqn. 17*. Define the *truncated transform* as:

$$U_T(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_T(s)\, e^{-ist} ds = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} u(s)\, e^{-ist} ds. \tag{36}$$

Define the *periodogram* ('average power') as:

$$\frac{1}{2T} \| U_T(t) \|^2. \tag{37}$$

Then for almost all $t \in \mathbb{R}$ and at every point of continuity of S_u, we have this limit relation between the antiderivative of the periodogram and the spectrum:

$$\lim_{T\to\infty} \int_0^t \frac{\| U_T(s) \|^2}{2T} ds = S_u(t). \tag{38}$$

If S_u is absolutely continuous (so that its derivative G_u, as given in *Eqn. 30*, exists almost everywhere), then at almost all $t \in \mathbb{R}$ and at each point of continuity of G_u:

$$\lim_{\varepsilon\to 0} \lim_{T\to\infty} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \frac{\| U_T(s) \|^2}{2T} ds = G_u(t). \tag{39}$$

If a function $f \in L^1$ has a *Fourier* transform F, then:

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{\| U_T(s) \|^2}{2T} F(s)\, ds = \int_{-\infty}^{\infty} G_u(t)\, F(t)\, dt. \tag{40}$$

Any one of *Eqns. 38–40* is the *Wiener–Khintchine* theorem, the not-so-trivial connection between correlation and power. Note that we do not have, as a pointwise almost everywhere limit:

$$G_u(t) = \lim_{T\to\infty} \frac{1}{2T} \| U_T(t) \|^2 \tag{41}$$

But *Eqn. 40* states precisely that *Eqn. 41* holds if interpreted in terms of generalized functions, while *Eqns. 38* and *39* are the integrated and differentiated forms, respectively. A necessary and sufficient condition for *Eqn. 41* to be true pointwise almost everywhere is an *open question* in mathematical analysis.

We have presented an adaptation of *Wiener*'s theory, defining autocorrelation and spectral density as a *Fourier* transform pair $\Gamma_u \leftrightarrow G_u$, whence the limit *Eqn. 41* (interpreted as in *Eqns. 38–40*) connecting spectral density and power follows as the *Wiener–Khintchine* theorem. *A. I. Khintchine* [17] used an alternate description (although his work was in the context of stochastic processes). In *Khintchine*'s theory, he began with truncated transforms *Eqn. 36*, and then defined G_u as the limit of the periodogram with *Eqn. 41*. Then, the fact that autocorrelation and spectral density turned out to be a *Fourier* transform pair, *i.e.*, that expressions *33–35* follow, is called the *Wiener–Khintchine* theorem. *Wiener*'s approach to this 'generalized harmonic analysis' is in no way probabilistic, and the theory applies to single well-defined functions rather than *Khintchine*'s ensembles of functions. We find *Wiener*'s approach more congenial.

8. Cross-Terms. – So far, we have considered only the autocorrelation-associated terms. Most of these definitions and relations have corresponding cross-term versions, the subject of this chapter. The (*time-average*) *correlation function* of $u, v \in W$ is Γ_{uv} : $\mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$\Gamma_{uv}(s) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t) \cdot v(s+t)\, dt. \tag{42}$$

The autocorrelation function defined in *Eqn. 13* is, thus, $\Gamma_u = \Gamma_{uu}$. One easily verifies the properties *q–s*:

q) $$\Gamma_{vu}(s) = \Gamma_{uv}(-s) \tag{43}$$

r) $$\Gamma_{uv}(0) = \langle u, v \rangle \tag{44}$$

s) $$|\Gamma_{uv}(s)|^2 \leq \Gamma_u(0)\, \Gamma_v(0), \tag{45}$$

but *not* necessarily $|\Gamma_{uv}(s)|^2 \leq |\Gamma_{uv}(0)|^2$ or $|\Gamma_{uv}(s)|^2 \leq \Gamma_u(s)\, \Gamma_v(s)$.
The *cross periodogram* is:

$$\frac{U_T(t) \cdot V_T(t)}{2T}. \tag{46}$$

The *cross spectral density* is:

$$G_{uv}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Gamma_{uv}(s)\, e^{-ist} ds, \tag{47}$$

with corresponding inversion theorem:

$$\Gamma_{uv}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G_{uv}(t)\, e^{ist} dt, \tag{48}$$

whence:

$$\Gamma_{uv} \leftrightarrow G_{uv}. \tag{49}$$

The 'cross' *Wiener–Khintchine* theorem is thus (with the appropriate functional-analytic assumptions as before):

$$G_{uv}(t) = \lim_{T\to\infty} \frac{U_T(t) \cdot V_T(t)}{2T}. \tag{50}$$

9. *Wiener* Description Space. – The space of type-(1,1) tensors may now be constructed from the *Hilbert* space W, as in [6]. Every element of $T_1^1(W)$ has a representation as a dyadic $a^i x_i$ with $a^i \in W^* = W$ and $x_i \in W$. We use a modified *Einstein* summation convention: the notation $a^i x_i$ implies an appropriate integer m and denotes the sum $\sum_{i=1}^{m} a^i x_i$.

The *double inner product* of $T_1^1(W)$, invariant with respect to the dyadic representation, is the double sum over i and j:

$$\langle\langle a^i x_i, b^j y_j\rangle\rangle = \langle a^i, b^j\rangle^* \langle x_i, y_j\rangle \tag{51}$$

The associated *norm* is:

$$\|a^i x_i\| = [\langle a^i, a^j\rangle^* \langle x_i, x_j\rangle]^{\frac{1}{2}}. \tag{52}$$

Note that the norm is *not* $\|a^i x_i\| = [\|a^i\|^{*2}\|x_i\|^2]^{\frac{1}{2}}$: one must take the 'cross-terms' into consideration, and so in *Eqn. 52* the summation is over both i and j. Since the *Hilbert* spaces W^* and W are isomorphic, henceforth we shall, for simplicity, drop the asterisks (*) in W^*, $\langle\cdot,\cdot\rangle^*$, and $\|\cdot\|^*$, but we shall keep the contravariant indices.

Let $a^1, a^2, \ldots, a^m$ be fixed in W, and let:

$$D = \{\mathbf{R} = a^i F_i : F_1, F_2, \ldots, F_m \in W\} \subset T_1^1(W). \tag{53}$$

Note that any of the a^i and F_i terms may be *constant* functions from $\mathbb{R}$ to $\mathbb{R}^3$, *i.e.*, $\mathbf{a}^i \in \mathbb{R}^3$ and $\mathbf{F}_i \in \mathbb{R}^3$. We call D the *Wiener description space* determined by the constitutive parameters $\{a^i\}$, and each $\mathbf{R} = a^i F_i \in D$ is a (*Wiener*) *response tensor*.

The correlation functions of the constitutive parameters are:

$$\Gamma_{a^i a^j}(s) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} a^i(t)\cdot a^j(s+t)\,dt. \tag{54}$$

Define the *phenomenological coefficients* as:

$$L^{ij} = \Gamma_{a^i a^j}(0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} a^i(t)\cdot a^j(t)\cdot dt = \langle a^i, a^j\rangle. \tag{55}$$

For $\mathbf{R} = a^i F_i \in D$ and $\mathbf{S} = a^j G_j \in D$, their double inner product is

$$\langle\langle \mathbf{R}, \mathbf{S}\rangle\rangle = \langle\langle a^i F_i, a^j G_j\rangle\rangle = L^{ij}\langle F_i, G_j\rangle. \tag{56}$$

It follows from Sect. 8 of [6] that the *Wiener* description space $(D, \langle\langle\cdot,\cdot\rangle\rangle)$ is a *Hilbert* space.

With the introduction of description spaces it is now possible to present a physical realization of the phenomenological calculus. An example that illustrates clearly the basic principles of the theory is the analysis of transport systems by irreversible thermodynamics. In brief, the F_i terms are the driving *forces*. As shown in [6], the (*Hilbert*-space) conjugate *flows* are, by the invariance of the response tensor of *Fig. 1*, given in the dual space as:

$$J^j = L^{ij} F_i. \tag{57}$$

The *Onsager* reciprocal relations,

$$L^{ij} = L^{ji} \tag{58}$$

follows from *Eqn. 55* and the symmetry:

$$\Gamma_{a^i a^j}(0) = \langle a^i, a^j\rangle = \langle a^j, a^i\rangle = \Gamma_{a^j a^i}(0). \tag{59}$$

For the response tensor $\mathbf{R}=a^iF_i\in D$, the squared norm is:

$$\delta = \|\mathbf{R}\|^2 = L^{ij}\langle F_i, F_j\rangle \tag{60}$$

which, for constant functions $\{\mathbf{F}_i\}$, becomes the familiar *dissipation* (*cf.* previous papers in our phenomenological calculus sequence)

$$\delta = L^{ij}\mathbf{F}_i\cdot\mathbf{F}_j. \tag{61}$$

The second law of thermodynamics is the statement

$$\delta \geq 0. \tag{62}$$

Thus the complete formalism of irreversible thermodynamics is given by the metric geometry associated with the response tensor.

10. Degree of Coupling. – The most important gain for our phenomenological calculus with the development of *Wiener* spaces is the extension of allowable constitutive parameters a^i to include nonconstant functions, hence random variables. Thus, it becomes possible to make stochastic models for the coupling coefficients L^{ij}, while at the same time to have steady-state forces.

It is important to note that, in contrast to the *Onsager* theory [18][19], the random variables a^i are in no way limited to thermal fluctuations from equilibrium, and hence the validity of description-space models is not restricted to states that are near equilibrium. Indeed, in *Wiener* description space we have an inherent measure of nearness-to-equilibrium. Note that *Onsager*'s reciprocity (*Eqn. 58*) is a consequence of the symmetry in *Eqn. 59* of the inner product of *Wiener* space *W*, whence $\Gamma_{a^ia^j}(0) = \Gamma_{a^ja^i}(0)$. For $s \neq 0$, we generally would have

$$\Gamma_{a^ia^j}(s) \neq \Gamma_{a^ja^i}(s). \tag{63}$$

When the correlation functions are continuous (*cf.* property *f* of Γ_u in *Chapt. 4*), however, as $s\rightarrow 0$, we have $\Gamma_{a^ia^j}(s) \rightarrow \Gamma_{a^ja^i}(s)$. Stated otherwise, when $s\approx 0$, we have $\Gamma_{a^ia^j}(s) \approx \Gamma_{a^ja^i}(s)$. So, variable s of the domain of the correlation function $\Gamma_{a^ia^j} : \mathbb{R} \rightarrow \mathbb{R}$ is a nearness-to-equilibrium indicator: small $|s|$ is 'near equilibrium', and large $|s|$ is 'far from equilibrium'. Thus, our phenomenological calculus encompasses *Onsager*'s stable and metastable realms as well as nonequilibrium dissipative structures.

In our paper [12] on the *Gibbs* paradox, it was posited that the identity of a given species was determined operationally by its transport properties. Thus, indentity was ultimately characterized by the constitutive parameter a^i. The degree to which two species couple is given by L^{ij}, which (by *Eqns. 55* and *42*) is seen to be an evaluation of the correlation function of a^i and a^j, *i.e.*, a measure of the similarity of the identity of species *i* and species *j*.

In [12], we mentioned that *Caplan*'s *degree of coupling* [20] is the dimensionless number *q*, defined by

$$q = \frac{L^{ij}}{\sqrt{L^{ii}L^{jj}}} \left(= \frac{\langle a^i, a^j \rangle}{\|a^i\| \, \|a^j\|} \right). \tag{64}$$

This ratio provides a measure of the efficiency of energy conversion in coupled flow processes such as biological transport systems, fuel cells, and desalination. When a^i is a fluctuation, *i.e.:*

$$a^i = \xi^i - \mu^i, \tag{65}$$

where ξ^i is a random variable with mean μ^i, *Eqn. 64* becomes

$$\rho = \frac{\langle a^i, a^j \rangle}{\|a^i\| \, \|a^j\|} = \frac{E\left(\left(\xi^i - \mu^i\right)\left(\xi^j - \mu^j\right)\right)}{\sigma_{\xi^i}\sigma_{\xi^j}}, \tag{66}$$

which is the *correlation coefficient* in statistics [21]. We see, therefore, that the degree of coupling q and the correlation coefficient ρ are phenomenologically the same entity, both being:

$$\frac{\langle a^i, a^j \rangle}{\|a^i\| \, \|a^j\|} = \frac{\Gamma_{a^i a^j}(0)}{\sqrt{\Gamma_{a^i}(0)\,\Gamma_{a^j}(0)}}. \tag{67}$$

The fact that this quantity has absolute value less than or equal to 1, *i.e.*,

$$-1 \leq q \leq +1, \tag{68}$$

is, of course, simply the *Schwarz* inequality in W (inequality *11*). In Euclidean spaces (or equivalently, when each a^i is a constant function), the degree of coupling is more commonly expressed as a cosine:

$$\cos\theta = q, \tag{69}$$

whence we may speak of *angle of coupling:*

$$\theta = \cos^{-1} q, \tag{70}$$

as we did in [12]. The bounds in *Eqn. 68* ensure that q falls in the domain of the arc-cosine function.

The degree of coupling concept may be extended in *Wiener* description space into the far-from-equilibrium realm. For a general $s \in \mathbb{R}$, define:

$$q(s) = \frac{\Gamma_{a^i a^j}(s)}{\sqrt{\Gamma_{a^i}(0)\,\Gamma_{a^j}(0)}}. \tag{71}$$

(The *Caplan* degree q in *Eqns. 64* and *67* is, hence, $q(0)$.) That

$$|q(s)| \leq 1 \tag{72}$$

is a consequence of the correlation property of inequality *45*. Thus $q : \mathbb{R} \rightarrow [0,1] \subset \mathbb{R}$.

11. Correlation in *D*. – Let $\{a^i\}$ be the set of constitutive parameters defining a *Wiener* description space D. The cross spectral densities of the constitutive parameters are the *Fourier* transforms of their correlation functions:

$$G_{a^i a^j}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Gamma_{a^i a^j}(s)\, e^{-ist} ds. \tag{73}$$

The truncated transforms of the constitutive parameters are:

$$A^i_T(t) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} a^i(s)\, e^{-ist} ds. \tag{74}$$

The *Wiener–Khintchine* theorem on the constitutive parameters of the description space D is thus (again with the appropriate functional-analytic interpretation):

$$G_{a^i a^j}(t) = \lim_{T\to\infty} \frac{A^i_T(t) \cdot A^j_T(t)}{2T}. \tag{75}$$

It relates the constitutive parameters and their *Fourier* transforms and correlations (*microscopic* factors) to the cross spectral power densities of the system (*macroscopic* factors).

The analysis may be carried out on a higher hierarchical level. It is implicit in *Eqn. 60* of $\|\mathbf{R}\|^2 = L^{ij}\langle F_i, F_j\rangle$ that for each of the m^2 pairs of indices (i, j) we have:

$$|L^{ij}| = |\langle a^i, a^j\rangle| \leq \|a^i\|\,\|a^j\| < \infty \tag{76}$$

and

$$|\langle F_i, F_j\rangle| \leq \|F_i\|\,\|F_j\| < \infty, \tag{77}$$

whence

$$\|\mathbf{R}\|^2 < \infty. \tag{78}$$

This says $\mathbf{R}$ satisfies the finite-limit condition *1*, and the *Wiener* theory of generalized harmonic analysis applies to the *Wiener* description space D as it does to the *Wiener* space W. The hierarchical extension is relatively trivial, except that one needs to be careful in the notational complications inherent in tensor analysis. We shall not further pursue the topic here, other than the item of correlation in D.

For $\mathbf{R} = a^i F_i \in D$ and $\mathbf{S} = a^j G_j \in D$, their correlation function resolves into components thus:

$$\Gamma_{\mathbf{RS}}(s) = \Gamma_{a^i a^j}(s)\, \Gamma_{F_i G_j}(s). \tag{79}$$

So, autocorrelation is:

$$\Gamma_{\mathbf{R}}(s) = \Gamma_{a^i a^j}(s)\, \Gamma_{F_i F_j}(s). \tag{80}$$

whence

$$\delta = \|\mathbf{R}\|^2 = \Gamma_{\mathbf{R}}(0) = \Gamma_{a^i a^j}(0)\, \Gamma_{F_i F_j}(0) = L^{ij}\langle F_i, F_j\rangle. \tag{81}$$

Eqn. 81 is another verification, although in an extremely complicated and round-about way, that the dissipation function δ is positive definite. This follows from the so-called *Bochner*'s theorem [22], which says that an autocorrelation function that is the limit of a *Cauchy–Riemann–Stieltjes* integral (limit integral *29*) of a distribution function is positive definite.

12. Epilogue. – It may be argued that harmonic analysis dates as far back as the Babylonian astronomers. Its 'modern' scientific beginning, however, is contained in *Jean Baptiste Joseph Fourier*'s discoveries published in his masterwork 'Théorie Analytique de la Chaleur' [23]. This book is still, after almost two centuries, the ultimate and indispensable source in the treatment of nearly every common and recondite topic in physics. *Fourier* first showed how any function might be represented by a trigonometric series, and he also advanced many methods and concepts that opened the way to modern mathematical physics.

Fourier analysis grew into 'harmonic analysis'. The term arose because *Fourier* began with periodic functions, and when expressed as a *Fourier* series, a periodic function displays its *harmonics*, multiples of the fundamental frequency. With the use of the *Fourier* transform, nonperiodic functions as well as periodic ones may be expressed in frequency domain. Through the study of the existence of the *Fourier* transform and its inverse, the subject of harmonic analysis evolved and developed. In particular, *generalized* harmonic analysis was introduced by *Wiener* [15] in this connection.

Harmonic analysis provides a powerful mathematical setting with which one explores the mysteries of the natural world *and* the formal world. It is, therefore, in this sense another categorical example of the modeling relation. With *Wiener* description space, our phenomenological calculus established connections to, and hence enters into the realm of, harmonic analysis.

Appendix. – *The Red House* (by *I. W. Richardson*). When I arrived at Dalhousie University in the summer of 1973, I was aware of unusual potential. Under the guidance of an enlightened administration, a small, provincial school had metamorphized into a progressive university, which was beginning to attract faculty with international reputations to fill its impressive new buildings. I was not aware that the Department of Physiology and Biophysics was famous across Canada for its group dynamics, which can best be described as pathological.

Potential became actual when, with the enthusiastic support of the President of the University and the Dean of Medicine, I was able to offer *Robert Rosen* a *Killam* professorship. *Bob* was bursting with ideas, and the *Killam* chair gave him the freedom and facilities to get them into print. I urged the Dean of Medicine to buy our fledging theoretical biology group one of the old houses lying along the few blocks separating the medical school from the university campus. He was very sympathetic to the need to be isolated from the incessant strife in the department, and bought a two-storey Victorian house painted bright red (*Fig. 2*).

The Red House was perfect for us. The front parlor made a sunlit, cheerful room for two secretaries. Well, it was sunlit on those rare days in Halifax when the sun shone, but, indeed, it was always cheerful. A large blackboard mounted on the wall transformed the dining room into a seminar room. With the installation of book shelves and good lights in the ceiling, the six bedrooms became dandy offices (*Fig. 3*) for graduate students and the three professors (*Herman Wolf*, a biophysicist, threw his lot in with us).

The heart of the Red House was the kitchen, where we met at coffee time and lunch. It was the scene of some pretty heavy scientific discussions, birthday celebrations, and warm camaraderie. In particular, I remember *Bob* trading warm, often poignant, stories with *Mary*, a university cleaning woman who regularly took her coffee breaks at the Red House. *Bob* told about growing up in New York City, and she

Fig. 2. *The Red House* (1975–1985). Photo by *A. H. Louie* (1981).

Fig. 3. Robert Rosen (left) *and* I. W. Richardson *in their Red House offices.* Photos by *A. H. Louie* (1981).

described life in a Newfoundland fishing village untouched by time. By his own example, *Bob* showed the students that *1*) doing science was exciting and lots of fun, and *2*) that doing science took discipline and lots of hard work. *Bob*'s capacity for work was awesome, and so was the volume of research he was publishing.

Early in 1982, it became certain that the Red House was doomed. Also, it became certain that I would go blind without the treatment available only at a major medical center. I left Dalhousie in September. *Bob* stayed on, and, in spite of failing health and frequent assaults upon the Red House, continued his efforts to forge a radically new philosophy of science. The Red House was eventually sold, and *Bob* was moved to an office in a house occupied by a group associated with the university. Then, he was moved again. In a letter of October 10, 1993, he wrote: '*I've been placed in complete exile in the Physical Plant building. That was the last straw, so I've put in for early retirement.*'

Robert Rosen's influence on theoretical biology was enormous and will grow as the critical re-evaluation of his work progresses.

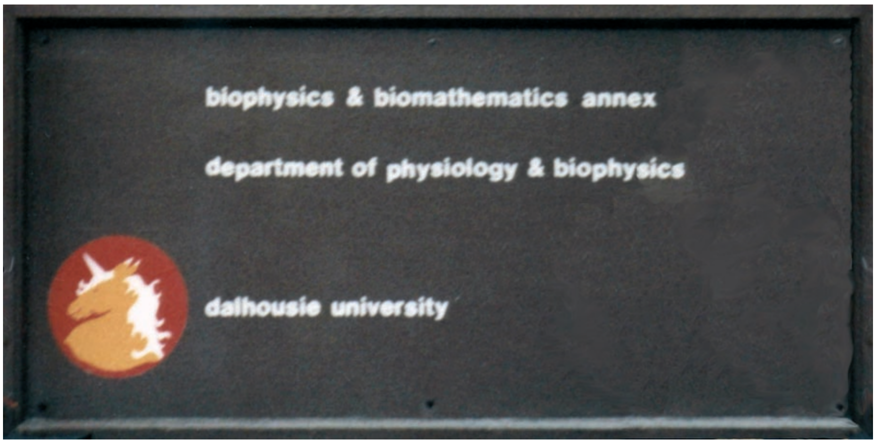

Fig. 4. *Our shingle.* Photo by *A. H. Louie* (1981).

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